2 Quadratic Equations

2.1 INTRODUCTION

A polynomial is an algebraic expression consisting of a finite number of terms with whole number exponents on the variables. Some examples of polynomials are

$$3x + 2,\ x^4 - 5x^2 + 3x + 4, \text{ and } 3x^2y - 6xy - 7.$$

The first two examples are called polynomials in x and the third is a polynomial in x and y. A polynomial containing exactly one term is called a monomial, one with exactly two terms is a binomial, and one with exactly three terms is a trinomial. Thus, $2x^2$ is a monomial, $2x^2 + 11x$ a binomial, and $2x^2 + 11x + 13$ a trinomial.

Algebraic expressions whose variables do not contain whole number exponents, such as $3x^{-2} + 7$ and $5x^{3/2} + 9x^{1/2} + 2$ are not polynomials.

Equation and Solution Sets

An equation is a statement indicating that two algebraic expressions are equal.

Examples of equations in x :
$2x - 3 = 5$ $x^2 - 3x = 2$ $2x + 3x = 5x$.

The third equation in the list is true no matter what value the variable x represents. An equation that is true for all real numbers for which both sides are defined is called an identity.
The first two equations in our list are true for only some numbers.
Solving an equation in x means determining all values of x which when substituted into the equation result in a true statement. Such values are called solutions or roots of the equation. These values are said to satisfy the equation. The set of all such solutions is called the equation's solution set.

For example, the equation $2x - 3 = 5$. becomes a true statement if we substitute 4 for x, resulting in $2(4) - 3 = 5$ (or $8 - 3 = 5$), a true statement. Since 4 is the only number that results in a true statement for this equation, the equation's solution set is {4}.

Linear equation

A linear or first-degree equation in one variable x is an equation that can be written in the standard form $ax + b = 0$
where a and b are real numbers and $a \neq 0$.
An example of a linear equation in one variable is $4x + 12 = 0$.
The solution is $x = -3$.

2.2 QUADRATIC EQUATION

An equation that contains a variable with an exponent of 2, but no higher power, is called a quadratic equation.

A quadratic equation in x is an equation that can be written in the standard form

$ax^2 + bx + c = 0$

where a, b and c are real numbers with $a \neq 0$. A quadratic equation in x is also called a second-degree polynomial equation in x.

Note In the definition of a quadratic equation, we must state $a \neq 0$. If we allowed a to become 0 in $ax^2 + bx + c = 0$ the resulting equation would be $bx + c = 0$ which is linear, and not quadratic.

Solving Quadratic Equations by Factoring

Factoring is the process of writing a polynomial as the product of two or more polynomials. We generally factor over the set of reals, meaning that the numerical coefficients in the factors are reals. For example,

$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$$

and $$\frac{1}{9}x^2 - 25 = \left(\frac{1}{3}x + 5\right)\left(\frac{1}{3}x - 5\right)$$

We say that $x^2 + 4$ cannot be factored using real coefficients. Such polynomials are irreducible over reals.
The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial's factors is irreducible. In this situation, the polynomial is said to be a factored completely.
If the trinomial $ax^2 + bx + c$ can be factored, then $ax^2 + bx + c = 0$ can be solved by using the zero product principle.

The zero product principle

Let A and B be real numbers, variables, or algebraic expressions. If AB = 0, then A = 0 or B = 0 or A and B are both 0. In words, this says that if a product is zero, then at least one of the factors is equal to zero.

STUDY TIP **Factoring $ax^2 + bx + c$ by Grouping**

An Example : $6x^2 + 19x - 7$

1. Multiply a and c $\quad 6(-7) = -42$
2. Find the factors of ac whose sum is b.
   21 and –2 are factors of –42 whose sum is 19.
3. Rewrite the middle term, bx as a sum or
   $6x^2 + 19x - 7$

   difference using the factors from step 2.

   $= 6x^2 + 21x - 2x - 7$
4. Factor by grouping.
   $= 3x(2x + 7) - 1(2x + 7)$
   $= (2x + 7)(3x - 1)$ or $(3x - 1)(2x + 7)$

The quadratic formula

The solution of the quadratic equation ,
$ax^2 + bx + c = 0$

is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Proof :

$ax + bx + c = 0$

Dividing by a we get $\quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

STUDY TIP

If we set b/2 = k, then the formula will take the form

$$x = \frac{-k \pm \sqrt{k^2 - ac}}{a}$$

This formula is advisable to be used in the cases when the coefficient b is an even number.

The Discriminant

The expression under the radical in the quadratic formula, $b^2 - 4ac = D$, is called the **discriminant**. The value of the discriminant determines whether the quadratic equation has two real solutions, one real solution, or no real solutions depending on whether D is positive, zero or negative. In the latter case, there will be two imaginary roots.

If D = 0, both roots reduce to $-\frac{b}{2a}$, and are thus equal to one another. In this case we do not say that the equation has only one root, but that it has two equal roots. It is clear that the roots will be unequal unless D = 0. When D = 0, the expression $ax^2 + bx + c$ is a perfect square of a linear expression in x.

Special Forms of Quadratic Equation

We will now consider some special forms of quadratic equations, in which one or more of the coefficients vanish.

A. If c = 0, the equation reduces to

$ax^2 + bc = 0,$

or $x(ax + b) = 0,$

the roots of which are 0 and $-\frac{b}{a}$.

B. If c = 0 and also b = 0, the equation reduces to $ax^2 = 0$, both roots of which are zero.

C. If b = 0, the equation reduces to $ax^2 + c = 0$ (called as **pure** quadratic equation), the roots of which are $\pm\sqrt{-\frac{c}{a}}$. The roots are therefore equal and opposite when b = 0, that is when the coefficient of x is zero.

D. If a, b and c are all zero, the equation is clearly satisfied for all values of x.

E If a and b are zero but c not zero,

put $x = \frac{1}{y}$ in the equation $ax^2 + bx + c = 0$;

then we have, after multiplying by y^2,

$cy^2 + by + a = 0.$

Now one root of this quadratic in y is zero if a = 0, and both roots are zero if a = 0 and also b = 0.

But since $x = \frac{1}{y}$, x is infinity when y is zero. Thus one root of $ax^2 + bx + c = 0$ is infinite if a = 0; also both roots are infinite if a = 0 and also b = 0.

Thus the quadratic equation

$(a - a')x^2 + (b - b')x + c - c' = 0$

has one root infinite, if a = a′; it has two roots infinite, if a = a′ and also b = b′; and the equation is satisfied for all values of x, if a = a′, b = b′ and c = c′.

Again, the equation

$a(x + b)(x + c) + b(x + c)(x + a) = c(x + a)(x + b),$

is a quadratic equation for all values of c except only when c = a + b, in which case the coefficient of x^2 in the quadratic equation is zero. When c = a + b we may still however consider that the equation is a quadratic equation, but with one of its roots infinite.

Note: It is however to be remarked that since infinite roots are not often of practical importance in Algebra, they are generally neglected unless specially required.

METHODS OF SOLVING A QUADRATIC EQUATION

Description and Form of the Quadratic Equation	Most Efficient Solution Method	Example
$ax^2 + c = 0$ The quadratic equation has no linear (x) term.	Solving for x^2 and the square root method	$4x^2 - 7 = 0$ $4x^2 = 7$ $x^2 = \frac{7}{4}$

		$x = \pm \frac{\sqrt{7}}{2}$
$(ax + b)^2 = d$	The square root method	$(x + 4)^2 = 5$ $x + 4 = \pm \sqrt{5}$ $x = -4 \pm \sqrt{5}$
$ax^2 + bx + c = 0$ where $ax^2 + bx + c$ can be easily factored.	Factoring and the zero product principle	$3x^2 + 5x - 2 = 0$ $(3x - 1)(x + 2) = 0$ $3x - 1 = 0$ or $x + 2 = 0$ $x = \frac{1}{3}$ $x = -2$
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ cannot be easily factored	The quadratic formula : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x^2 - 2x - 6 = 0$ $x = \frac{2 \pm \sqrt{4 - 4(1)(-6)}}{2(1)}$ $= \frac{2 \pm \sqrt{28}}{2}$ $= 1 \pm \sqrt{7}$

A quadratic equation cannot have more than two roots.

For, if possible, let the equation $ax^2 + bx + c = 0$ have three different roots α, β, γ. Then since each of these values must satisfy the equation, we have

$a\alpha^2 + b\alpha + c = 0,$ ...(1)

$a\beta^2 + b\beta + c = 0,$ ...(2)

$a\gamma^2 + b\gamma + c = 0.$ ...(3)

From (1) and (2), by substraction,

$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$;

dividing by $\alpha - \beta$ which is not zero, we get ;

$a(\alpha + \beta) + b = 0$

Similarly from (2) and (3)

$a(\beta + \gamma) + b = 0$;

$\therefore$ by subtraction

$a(\alpha - \gamma) = 0$;

which is impossible, since a is not zero.

➤ **Example 1.** Solve the equation $x^2 - 7x + 12 = 0$

➤ ***Solution*** Let x_1 and x_2 be roots of the given equation . Then $x_1x_2 = 12$, that is x_1 and x_2 are numbers having the same sign. Further, $x_1 + x_2 = 7$ and, hence, x_1 and x_2 are positive numbers. Thus, we need to find two positive numbers whose sum is equal to 7, and the product to 12. It is easy to guess that these numbers are 3 and 4. Thus, the roots of the given equation are $x_1 = 3, x_2 = 4$.

➤ **Example 2.** Solve the equation $(x - 1)(x-2)(x-3)(x-4) = 3$.

➤ ***Solution*** We have $(x - 1)(x - 4) = x^2 - 5x + 4$ and $(x - 2)(x - 3) = x^2 - 5x + 6$. The given equation can be rewritten as follows :

$$(x^2 - 5x + 4)(x^2 - 5x + 6) = 3.$$

We now introduce a new variable setting

$$x^2 - 5x + 4 = y;$$

note that $x^2 - 5x + 6 = (x^2 - 5x + 4) + 2 = y + 2.$

We obtain

$$y(y+2)=3 \Rightarrow y^2+2y-3=0 \Rightarrow y_1=-3,\ y_2=1.$$

It remains now to solve the following two equations:

$$x^2-5x+4=-3,\quad x^2-5x+4=1.$$

The first equation has no solution, while from the second equation we find $x_1=(5+\sqrt{13})/2$, $x_2=(5-\sqrt{13})/2$.

STUDY TIP Whenever an equation contains fractions in whose denominators the unknown quantity occurs, the equation may be multiplied by the lowest common multiple of the denominators.

➤ **Example 3.** Solve the equation $\dfrac{3}{x-5}+\dfrac{2x}{x-3}=5.$

➤ ***Solution*** Multiply by $(x-5)(x-3)$, the L.C.M. of the denominators; then we have

$$3(x-3)+2x(x-5)=5(x-5)(x-3);$$

$$\therefore\quad 3x^2-33x+84=0.$$

$$x=4 \text{ or } x=7.$$

➤ **Example 4.** Solve the equation

$$\frac{2}{2-x}+\frac{1}{2}-\frac{4}{x(2-x)}=0$$

➤ ***Solution*** This equation contains the variable in the denominator. Let us transpose all the terms of the given equation into a common fraction .

$$\frac{2}{2-x}+\frac{1}{2}-\frac{4}{x(2-x)}=0;$$

$$\frac{4x+x(2-x)-8}{x(2-x)}=0;\qquad \frac{x^2-6x+8}{x(2-x)}=0.$$

Since the fraction is equal to zero, we get

$$\begin{cases} x^2-6x+8=0 \\ x(2-x)\neq 0 \end{cases}.$$

From the equation $x^2-6x+8=0$, we find : $x=2$, $x=4$. The statement $2(2-2)\neq 0$ is false; hence, 2 is not a root of the given equation (since for $x=2$ the denominator vanishes). The statement $2(4-2)\neq 0$ is true; hence, 4 is the only root of the given equation.

➤ **Example 5.** Solve the equation

$$\frac{x^2-3x}{x^2-1}+2+\frac{1}{x-1}=0.$$

➤ ***Solution*** Multiply by x^2-1, the L.C.M. of the denominators; then we have

$$x^2-3x+2(x^2-1)+x+1=0,$$

which reduces to $3x^2-2x-1=0$,

that is $(3x+1)(x-1)=0$.

Thus the roots appear to be $-\dfrac{1}{3}$ and 1; the latter root is however due to the multiplication by x^2-1.

Since

$$\frac{x^2-3x}{x^2-1}+\frac{1}{x-1}=\frac{x^2-3x+x+1}{x^2-1}=\frac{(x-1)^2}{x^2-1}=\frac{x-1}{x+1},$$

the equation is equivalent to

$$\frac{x-1}{x+1}+2=0.$$

which has only one root, namely $x=-\dfrac{1}{3}$.

CAUTION From the above example it will be seen that when an equation has been made integral by multiplication, some of the roots of the resulting equation may have to be rejected.

Practice Problems

1. Solve for x in the equation
   $2m(1+x^2)-(1+m^2)(x+m)=0.$
2. Find the roots of the equation
   $(b-c)x^2+(c-a)x+(a-b)=0.$
3. Solve the equation $(x^2-3x)^2+3x^2-9x-28=0$ for real roots.
4. Solve the equation $\left(\dfrac{x-1}{x}\right)^2-3\left(\dfrac{x-1}{x}\right)-4=0.$

5. Solve the equation $1-\dfrac{3-2x}{5-x}=\dfrac{3}{3-x}-\dfrac{x+3}{x+1}$.

2.3 SUM AND PRODUCT OF ROOTS

Vieta's Theorem

If α & β are the roots of the quadratic equation $ax^2+bx+c=0$, then

(i) $\alpha+\beta=-b/a$ (ii) $\alpha\beta=c/a$

Proof :

Let the quadratic equation $ax^2+bx+c=0$ be written as

$$\frac{1}{a}(ax^2+bx+c)=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad ...(1)$$

If α & β are the roots of the quadratic equation $ax^2+bx+c=0$ then

$$(x-\alpha)(x-\beta)=0$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0 \qquad ...(2)$$

Comparing the coefficients of similar powers of x in (1) & (2) we have

$$\text{sum of roots}=\alpha+\beta=-\frac{b}{a}$$

$$\text{product of roots}=\alpha\beta=\frac{c}{a}$$

Note

$$|\alpha-\beta|=\sqrt{(\alpha-\beta)^2}=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$=\sqrt{\frac{b^2-4ac}{a^2}}=\frac{\sqrt{D}}{|a|}$$

Formation of Quadratic Equation

Although we cannot in all cases find the roots of a given equation, it is very easy to solve the converse problem, namely the problem of finding an equation which has given roots.

For example, to find the equation whose roots are 4 and 5.

We want to find an equation which is satisfied when $x = 4$, or when $x = 5$; that is when $x - 4 = 0$, or when $x - 5 = 0$; and in no other cases. The equation required must be

$$(x-4)(x-5)=0,$$

that is,

$$x^2-9x+20=0.$$

Suppose α and β are the roots of a quadratic equation

$$ax^2+bx+c=0$$

Then, we have

$$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$

$$=x^2-(\text{sum of roots})x+(\text{product of roots})$$

$$=x^2+\frac{b}{a}x+\frac{c}{a}=\frac{1}{a}(ax^2+bx+c).$$

This leads to the factorization,

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

Thus, if we know the roots of a quadratic equation, then we can write down a factorization of the corresponding quadratic function.

Conversely, if α and β are two numbers (real or complex), then the most general quadratic equation having α and β as its roots is given by

$$a(x-\alpha)(x-\beta)=0,$$

where a is a non-zero, real or complex number. If we restrict the coefficient of x^2 to be 1 then we get a unique quadratic equation

$$(x-\alpha)(x-\beta)=0.$$

This is the same as $x^2-(\alpha+\beta)x+\alpha\beta=0$.

STUDY TIP A quadratic polynomial in which the coefficient of x^2 is 1 is called a monic quadratic polynomial (or a reduced quadratic).

➤ **Example 1.** Set a quadratic equation with the given numbers as its roots 7 and –3

➤ ***Solution*** We shall seek the desired equation in the form $x^2 - px + q = 0$. We have $p = (7 - 3) = 4$, $q = 7(-3) = -21$; then $x^2 - 4x - 21 = 0$.

➤ **Example 2.** A quadratic polynomial p(x) has $1 + \sqrt{5}$ and $1 - \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find the quadratic polynomial.

➤ ***Solution*** sum of the roots = 2

product of the roots = – 4

$\therefore$ let $p(x) = a(x^2 - 2x - 4)$

$p(1) = 2$

$\Rightarrow \quad 2 = a(1^2 - 2 \cdot 1 - 4)$

$\Rightarrow \quad a = -2/5$

$\therefore \quad p(x) = -2/5\,(x^2 - 2x - 4)$

➤ **Example 3.** Let α and β be roots of the equation $2x^2 - 3x - 7 = 0$. Without computing α and β, set a quadratic equation with α/β and β/α as its roots.

➤ ***Solution*** We shall seek the required equation in the form $x^2 - px + q = 0$, then

$$p = \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$q = \frac{\alpha}{\beta}\,\frac{\beta}{\alpha} = 1.$$

Since α and β are roots of the equation $2x^2 - 3x - 7 = 0$,

$$\alpha\beta = -\frac{7}{2}, \alpha + \beta = \frac{3}{2}.$$

$$\alpha^2 + \beta^2 = \left(\frac{3}{2}\right)^2 - 2\left(-\frac{7}{2}\right) = \frac{9}{4} + 7 = \frac{37}{4}$$

hence, $$p = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{37/4}{-7/2} = -\frac{37.2}{4.7} = -\frac{37}{14}$$

Thus, the coefficients p and q of the desired quadratic equation have been found : $p = -37/14$, $q = 1$. The equation has the form

$$x^2 + \frac{37}{14}x + 1 = 0, \text{ or } 14x^2 + 37x + 14 = 0.$$

➤ **Example 4.** If α and β are the roots of the equation $x^2 + \sqrt{2}\,x + 3 = 0$, find the monic quadratic having zeros

$\alpha^2 + \beta^2$ and $2\alpha\beta$.

➤ ***Solution*** The monic quadratic having $\alpha^2 + \beta^2$ and $2\alpha\beta$ as its roots is given by

$$x^2 - (\alpha^2 + \beta^2 + 2\alpha\beta)\,x + 2\alpha\beta(\alpha^2 + \beta^2). \quad ...(1)$$

However, we know that

$$\alpha + \beta = -\sqrt{2}, \alpha\beta = 3.$$

Hence, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 2 - 6 = -4$

This gives, $\alpha^2 + \beta^2 + 2\alpha\beta = 2$

and $2\alpha\beta\,(\alpha^2 + \beta^2) = -24$.

Hence, from (1) the required quadratic is $x^2 - 2x - 24 = 0$

➤ **Example 5.** The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Find the value of $\frac{n}{p}$.

➤ ***Solution*** $x^2 + mx + n = 0$ < 2α, 2β

and $x^2 + px + n = 0$ < α, β

$2(\alpha + \beta) = -m$ ....(1)

$4\alpha\beta = n$ ....(2)

and $\alpha + \beta = -p$ ....(3)

$\alpha\beta = m$ ....(4)

$\therefore$ (1) and (3) $\Rightarrow$ $2p = m$

and (2) and (4) $\Rightarrow$ $4m = n$

$$\Rightarrow \quad \frac{n}{p} = \frac{4m}{m/2} = 8$$

➤ **Example 6.** Find the values of a for which one of the roots of $x^2 + (2a + 1)\,x + (a^2 + 2) = 0$ is twice the other root. Find also the roots of this equation for these values of a.

➤ ***Solution*** $\alpha + 2\alpha = -(2a + 1)$

and $2\alpha^2 = a^2 + 2$.

The first relation gives $\alpha = \dfrac{-(2a+1)}{3}$

Substituting this value of α in the second relation, we get

$$2(2a + 1)^2 = 9(a^2 + 2).$$

This is the same as

$$a^2 - 8a + 16 = 0.$$

Thus we get a quadratic equation for a and this equation has coincident roots a = 4. Thus there is a unique value of a for which the conditions of the problem are fulfilled. Corresponding to this value of a, the given equation reduces to

$$x^2 + 9x + 18 = 0$$

Now $x^2 + 9x + 18 = (x + 6)(x + 3)$

so that the roots are $\alpha = -6$, $\beta = -3$.

➤ **Example 7.** Let α, β be the roots of $ax^2 + bx + c = 0$, $\alpha_1, -\beta$ be the roots of $a_1x^2 + b_1x + c_1 = 0$ then find a quadratic equation whose roots are α, α_1.

➤ ***Solution*** Given $\alpha + \beta = -b/a$ ...(1)

$\alpha\beta = c/a$ ...(2)

Also $\alpha_1 - \beta = -b_1/a_1$ ...(3)

$-\alpha_1\beta = c_1/a_1$ ...(4)

Now $(\alpha + \beta) + (\alpha_1 - \beta_1) = -\dfrac{b}{a} - \dfrac{b_1}{a_1}$

$$\Rightarrow \quad \alpha + \alpha_1 = -\frac{b}{a} - \frac{b_1}{a_1} \quad ...(5)$$

Form (1) & (2),

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c} \quad ...(6)$$

Similarly from (3) & (4),

$$\frac{1}{\alpha_1} - \frac{1}{\beta} = \frac{-b_1}{c_1} \quad ...(7)$$

From (6) + (7),

$$\frac{1}{\alpha} + \frac{1}{\alpha_1} = -\frac{b}{c} - \frac{b_1}{c_1}$$

$$\Rightarrow \quad \frac{\alpha+\alpha_1}{\alpha\alpha_1} = \frac{-b}{c} - \frac{b_1}{c_1} \quad ...(8)$$

Now the equation whose roots are α, α_1 is $x^2 - (\alpha + \alpha_1)x + \alpha\alpha_1 = 0$

$$\Rightarrow \quad \frac{x^2}{\alpha+\alpha_1} - x + \frac{\alpha\alpha_1}{\alpha+\alpha_1} = 0$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{b}{a} + \frac{b_1}{a_1}\right)} + x + \frac{1}{\left(\frac{b}{c} + \frac{b_1}{c_1}\right)} = 0$$

➤ **Example 8.** If α is a root of $4x^2 + 2x - 1 = 0$ prove that $4\alpha^3 - 3\alpha$ is the other root.

➤ ***Solution*** Let the other root be β then

$$\alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$$

$$\Rightarrow \quad \beta = -\frac{1}{2} - \alpha \quad ...(1)$$

and $4\alpha^2 + 2\alpha - 1 = 0$, because α is a root of $4x^2 + 2x - 1 = 0$.

Now $4\alpha^3 - 3\alpha = \alpha(4\alpha^2 - 3)$

$= \alpha(1 - 2\alpha - 3)$ $\{\text{since } 4\alpha^2 + 2\alpha - 1 = 0\}$

$= -2\alpha^2 - 2\alpha = -\dfrac{1}{2}(4\alpha^2) - 2\alpha = -\dfrac{1}{2}(1 - 2\alpha) - 2\alpha$

$= -\dfrac{1}{2} - \alpha = \beta$ $\{\text{from (1)}\}$

Practice Problems

1. Suppose the sum of the roots of $ax^2 - 6x + c = 0$ is -3 and their product is 2 , find the values of a and c.
2. Given that α and β are the roots of $6x^2 - 5x - 3 = 0$ find the reduced quadratic equation whose roots are $\alpha - \beta^2$ and $\beta - \alpha^2$.

3. Find a quadratic in x which is divisible by $x - 2$ and assumes value -7 and 3 for $x = 1$ and -1 respectively.

4. Find the values of m, for which the equation $5x^2 - 24x + 2 + m(4x^2 - 2x - 1) = 0$ has
(a) equal roots
(b) the product of the roots is 2
(c) the sum of the roots is 6

5. Suppose x_1 and x_2 are roots of the equation $x^2 + x - 7 = 0$. Find
(a) $x_1^2 + x_2^2$
(b) $x_1^3 + x_2^3$
(c) $x_1^4 + x_2^4$ without solving the equation.

6. If one root of $k(x - 1)^2 = 5x - 7$ is double the other root, show that k is either 2 or -25.

7. For what values of a is the ratio of the roots of the equation $x^2 + ax + a + 2 = 0$ equal to 2 ?

8. For what value of a is the difference between the roots of the equation $(a - 2)x^2 - (a - 4)x - 2 = 0$ equal to 3 ?

9. If the difference of roots of the equation $2x^2 - (a + 1)x + a - 1 = 0$ is equal to their product, then prove that $a = 2$.

10. Two candidates attempt to solve a quadratic equations of the form, $x^2 + px + q = 0$. One starts with a wrong value of 'p' and finds the roots to be 2 & 6. The other starts with a wrong value of 'q' and finds the roots to be 2 & -9. Find the correct roots.

11. If α and β are the roots of the equation $x^2 + px + q = 0$, then show that α/β is a root of the equation $qx^2 - (p^2 - 2q)x + q = 0$.

12. Find $b \in I$ for the equation $5x^2 + bx - 28 = 0$ if the roots α and β satisfy $5\alpha + 2\beta = 1$.

13. If the roots of the equations $ax^2 + bx + c = 0$ are $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, prove that $(a + b + c)^2 = b^2 - 4ac$.

14. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and γ, δ those of equation $lx^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

15. Let p(x) be a quadratic polynomial such that for distinct reals α and β, $p(\alpha) = \alpha$, $p(\beta) = \beta$ show that α and β are roots of $p[p(x)] - x = 0$.

2.4 IDENTITY

An **identity** in x is satisfied by all permissible values of x, where as an **equation** in x is satisfied by some particular value of x. For example, $(x + 1)^2 = x^2 + 2x + 1$ is an identity in x. It is satisfied for all values of x.

A quadratic equation is satisfied exactly two values of 'x' which may be real or imaginary. We should note that the quadratic equation $ax^2 + bx + c = 0$ cannot have more than two different roots, unless $a = b = c = 0$; and when a, b, c are all zero it is clear that the equation $ax^2 + bx + c = 0$ will be satisfied for all values of x, that is to say the equation is an identity.If a quadratic equation is satisfied by three or more distinct values of 'x', then it is an identity.

Note

1. Two equations are said to be **equivalent** if they have the same roots.

2. Two equations in x are **identical** if and only if the coefficient of similar powers of x in the two equations are proportional. Identical equations have the same roots.

3. An equation remains unchanged if it is multiplied or divided by a non-zero number.

Thus, $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ are identical, if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$

$x^2 - 5x + 4 = 0$ and $2x^2 - 10x + 8 = 0$ are identical equations.

Both these equations have same roots 1 and 4.

STUDY TIP The equation $ax^2 + bx + c = 0$ is :

a quadratic equation if	$a \neq 0$	Two Roots
a linear equation if	$a = 0, b \neq 0$	One Root
a contradiction if	$a = b = 0, c \neq 0$	No Root
an identity if	$a = b = c = 0$	Infinite Roots

➤ **Example 1.** Solve the equation

➤ ***Solution***

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2$$

The equation is satisfied is satisfied by x = a, by x = b, or by x = c. Since it is only of the second degree in x, it must be an identity.

➤ **Example 2.** Solve the equation a^2

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}=x^2.$$

➤ ***Solution*** The equation is clearly satisfied by x = a, and also by x = b; hence a, b are roots of the equation, and these are the only roots of the quadratic equation. The equation is not an identity, for it is not satisfied by x = c.

➤ **Example 3.** If $a+b+c=0$; $an^2+bn+c=0$ and $a+bn+cn^2=0$ where $n \neq 0, 1$, then prove that $a = b = c = 0$.

➤ ***Solution*** Note that $ax^2 + bx + c = 0$ is satisfied by

$$x = 1\ ;\ x = n\ \&\ x = \frac{1}{n} \quad \text{where} \quad n \neq \frac{1}{n}$$

⇒ The quadratic equation has 3 distinct real roots which implies that it must be an identity.

➤ **Example 4.** Show that

$$\frac{(x+b)(x+c)}{(b-c)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1$$

is an identity.

➤ ***Solution*** The given equation is quadratic equation in x.

put $x = -a$ in the given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(b-a)(c-a)}{(b-a)(c-a)} = 1 = \text{R.H.S.}$$

Put $x = -b$ in given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(c-b)(a-b)}{(c-b)(a-b)} = 1 = \text{R.H.S.}$$

Put $x = -c$ in given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1 = \text{R.H.S.}$$

So, given equation is satisfied by 3 values. So it is an identity.

➤ **Example 5.** Show that values of a, b, c exist such that $2x^2 - 5x - 1 \equiv a(x + 1)(x - 2) + b(x - 2)(x - 1) + c(x - 1)(x + 1)$ and find these values.

➤ ***Solution*** The right-hand side

$= a(x^2 - x - 2) + b(x^2 - 3x + 2) + c(x^2 - 1)$

$= (a + b + c)x^2 - (a + 3b)x - (2a - 2b + c).$

This is equal to the left-hand side for all values of x, if values for a, b, c can be found such that

$a + b + c = 2,\ a + 3b = 5,\ 2a - 2b - c = 1.$

These equation have one solution, namely,

$a = 2,\ b = 1,\ c = -1.$

STUDY TIP In a question of this kind if it has been shown that an identity of the specified form exists, the values of the constants may be found by giving special values to the variables.

Thus, assuming that an identity of the specified form exists, we can find a, b, c, by putting x = 1, –1, 2 in succession.

2.5 TRANSFORMATION OF EQUATION

From the equation $f(x) = 0$ we may form an equation whose roots are connected with those of the given equation by some assigned relation.

Let y be a root of the required equation and let $\phi(x, y) = 0$ denote the assigned relation ; then the transformed equation can be obtained either by expressing x as a function of y by means of the equation $\phi(x, y) = 0$ and substituting this value of x in $f(x) = 0$; or by eliminating x between the equations $f(x) = 0$ and $\phi(x, y) = 0$.

STUDY TIP Let α, β be the roots of $ax^2 + bx + c = 0$, and suppose that we require the equation whose roots are a uniform function of α, β say $\phi(\alpha), \phi(\beta)$, where f(x) is a given function of x.
Let $y = \phi(x)$ and suppose that from this equation we can find x in terms of y,
which we denote by $x = \phi^{-1}(y)$.
Transforming the equation $ax^2 + bx + c = 0$ by the substitution $x = \phi^{-1}(y)$, we obtain a new equation in y, which contains the roots $\phi(\alpha), \phi(\beta)$.

To find the equation whose roots are the squares of those of a proposed equation.

Let $f(x) = 0$ be the given equation ; putting $y = x^2$, we have $x = \sqrt{y}$; hence the required equation is $f(\sqrt{y}) = 0$.

➤ **Example 1.** Find the equation whose roots are the squares of those of the equation $x^2 + ax + b = 0$.

➤ ***Solution*** Putting $x = \sqrt{y}$, and transposing, we have

$$(y + b) = -a\sqrt{y} \;;$$

whence $\quad y^2 + 2by + b^2 = a^2y$,

or $\quad y^2 + (2b - a^2)y + b^2 = 0$

To transform an equation into another whose roots exceed those of the proposed equation by a given quantity.

Let $f(x) = 0$ be the proposed equation, and let h be the given quantity ; put $y = x + h$, so that $x = y - h$; then the required equation is $f(y - h) = 0$.

Similarly $f(y + h) = 0$ is an equation whose roots are less by h than those of $f(x) = 0$.

➤ **Example 2.** Remove the second term from the equation $px^2 + qx + r = 0$

➤ ***Solution*** Let α, β be the roots, so that $\alpha + \beta = -\frac{q}{p}$. The if we increase each of the roots by $\frac{q}{2p}$, in the transformed equation the sum of the roots will be equal to $-\frac{q}{p} + \frac{q}{p}$; that is, the coefficient of the second term will be zero.

Hence the required transformation will be effected by substituting $x - \frac{q}{2p}$ for x in the given equation.

$$p\left(x - \frac{q}{2p}\right)^2 + q\left(x - \frac{q}{2p}\right) + r = 0$$

➤ **Example 3.** Let α and β be the roots of $x^2 - 3x + 1 = 0$. Find a quadratic equation whose roots are $\frac{\alpha}{\alpha - 2}$ and $\frac{\beta}{\beta - 2}$. Hence or otherwise find the value of $\frac{\alpha}{\alpha - 2} \cdot \frac{\beta}{\beta - 2}$.

➤ ***Solution*** Let $y = \frac{x}{x - 2}$ where x can take values α, β

$$\therefore \quad xy - 2y = x \;\Rightarrow\; x = \frac{2y}{y - 1} \qquad ....(1)$$

substituting the value of x in $x^2 - 3x + 1 = 0$ and simplifying we get the required quadratic as

$$y^2 - 4y - 1 = 0 \qquad ....(2)$$

now from (2) $\frac{\alpha}{\alpha - 2} \cdot \frac{\beta}{\beta - 2} = -1$ using product of roots.

➤ **Example 4.** If $\tan\alpha$, $\tan\beta$ are the roots of $x^2 - px + q = 0$ and $\cot\alpha$, $\cot\beta$ are the roots of $x^2 - rx + s = 0$ then find the value of *rs* in terms of *p* and *q*.

➤ ***Solution*** $x^2 - px + q = 0$ with roots $\tan\alpha$, $\tan\beta$ .....(1);

$x^2 - rx + s = 0$ with roots $\cot\alpha$, $\cot\beta$ ....(2)

Here roots of (2) are reciprocal of (1)

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$\therefore$ put x as $\frac{1}{y}$ in (2) $\Rightarrow \frac{1}{y^2} - \frac{r}{y} + s = 0$

$\Rightarrow$ $sy^2 - ry + 1 = 0$

We can replace y by x to get an equation in x,

$sx^2 - rx + 1 = 0$ ....(3)

Comparing (1) and (3)

$$\frac{1}{s} = \frac{p}{r} = q \quad s = \frac{1}{q}; \quad r = \frac{p}{q}$$

hence $rs = \frac{p}{q^2}$

Practice Problems

1. If $(p^2 - 1)x^2 + (p - 1)x + p^2 - 4p + 3 = 0$ be an identity in x, then find the value of p.
2. Find all polynomials p satisfying $p(x + 1) = p(x) + 2x + 1$.
3. If n is any number, show that n^2 can be expressed in the form $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and find the values of a, b, c.
4. Show that constants a, b, c can be found such that $3x^2 + 16x + 23 \equiv a(x + 2)(x + 3) + b(x + 3)(x + 1) + (x + 1)(x + 2)$ and find the values of a, b, c.
5. If a, b, c are distinct real numbers, then solve for x:
$$\frac{(a+x)^2}{(a-b)(a-c)} + \frac{(b+x)^2}{(b-c)(b-a)} + \frac{(c+x)^2}{(c-a)(c-b)} = 1.$$
6. Solve the equation, $\frac{x-ab}{a+b} + \frac{x-bc}{b+c} + \frac{x-ca}{c+a} = a + b + c$. What happens if $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$
7. If α, β are the roots of $2x^2 - 5x - 4 = 0$, find the simplest quadratic equation whose roots are $\alpha + 1/\alpha, \beta + 1/\beta$.
8. Prove that the equation whose roots are squares of the roots of the equation $ax^2 + bx + c = 0$ is $a^2x^2 + (2ac - b^2)x + c^2 = 0$
9. If α and β are roots of the equation $x^2 - 2x + 3 = 0$, find the equation whose roots are
(i) $\alpha + 2, \beta + 2$ (ii) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$
10. If α and β are roots of the equation $ax^2 + bx + c = 0$, find the roots of the equation
(i) $ac\,x^2 - (b^2 - 2ac)x + ac = 0$
(ii) $acx^2 + b(a + c)x + (a + c)^2 = 0$ in terms of α and β.

2.6 SYMMETRIC FUNCTIONS OF ROOTS

By the symmetric function of roots we mean those functions which remain unaltered in value when any two of the roots are interchanged. Without knowing the separate values of the roots in terms of the coefficients, we can calculate the values of symmetric functions of roots in terms of the coefficients. For example, if α, β are roots of quadratic equation $ax^2 + bx + c = 0$, we can find out the value of

(i) $\alpha + \beta + \alpha\beta$, (ii) $\alpha^2\beta + \beta^2\alpha$

in terms of a, b and c since each of these relations are symmetric functions of the roots, ase there will be no change in them if α and β are interchanged.

To find out the value of the above symmetric functions, we take the help of the following relations, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ and some important formulae which are given below :

STUDY TIP In order to find the value of symmetric function of α and β, express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. For this use the following results whichever is applicable.

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha.\beta$

(ii) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(iii) $\alpha^4+\beta^4 = (\alpha^3+\beta^3)(\alpha+\beta) - (\alpha\beta^3+\alpha^3\beta)$

(iv) $(\alpha+\beta)^4 = (\alpha^4+6\alpha^3\beta+4\alpha^2\beta^2+6\alpha\beta^3+\beta^4$
$= \alpha^4+\beta^4+6\alpha\beta(\alpha^2+\beta^2)+4\alpha^2\beta^2$

(v) $\alpha^5+\beta^5 = (\alpha^3+\beta^3)(\alpha^2+\beta^2) - (\alpha^2\beta^3+\alpha^3\beta^2)$
$= (\alpha^3+\beta^3)(\alpha^2+\beta^2) - \alpha^2\beta^2(\alpha+\beta)$

➤ **Example 1.** If α and β are the roots of $x^2+4x+6=0$, then find the value of

(i) $1/\alpha + 1/\beta$ (ii) $\alpha^3+\beta^3$ (iii) $\alpha/\beta+\beta/\alpha$

we have $\alpha+\beta = -4$, $\alpha\beta = 6$.

➤ ***Solution*** Hence $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-4}{6} = \frac{-2}{3}$.

We can write

$$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = -64+72 = 8.$$

And lastly, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$$= \frac{(-4)^2-(2\times 6)}{6} = 4/6 = 2/3.$$

➤ **Example 2.** Let α and β be the roots of the quadratic equation $(x-2)(x-3)+(x-3)(x+1)+(x+1)(x-2)=0$. Find the value of

$$\frac{1}{(\alpha+1)(\beta+1)} + \frac{1}{(\alpha-2)(\beta-2)} + \frac{1}{(\alpha-3)(\beta-3)}.$$

➤ ***Solution*** Quadratic equation is

$$3x^2-8x+1=0 \quad \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$$

$\alpha+\beta = \frac{8}{3}$ and $\alpha\beta = \frac{1}{3}$

to find the value of

$$\frac{1}{\alpha\beta+(\alpha+\beta)+1} + \frac{1}{\alpha\beta-2(\alpha+\beta)+4} + \frac{1}{\alpha\beta-3(\alpha+\beta)+9}$$

$$= \frac{1}{4} - 1 + \frac{3}{4} = -\frac{3}{4} + \frac{3}{4} = 0$$

➤ **Example 3.** Let p & q be the two roots of the equation, $mx^2+x(2-m)+3=0$. Let m_1, m_2 be the two values of m satisfying $\frac{p}{q}+\frac{q}{p} = \frac{2}{3}$. Determine the numerical value of $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}$.

➤ ***Solution*** $mx^2+(2-m)x+3=0 \quad \begin{matrix} \nearrow p \\ \searrow q \end{matrix}$

$$p+q = \frac{m-2}{m} \ ; \ pq = \frac{3}{m}$$

now m_1 and m_2 satisfies $\frac{p}{q}+\frac{q}{p} = \frac{2}{3}$

$$\Rightarrow \frac{p^2+q^2}{pq} = \frac{2}{3}$$

$$\frac{(p+q)^2-2pq}{pq} = \frac{2}{3}$$

$$\left(\frac{m-2}{m}\right)^2 - \frac{6}{m} = \frac{2}{3}.\frac{3}{m} = \frac{2}{m}$$

$$\Rightarrow \frac{(m-2)^2}{m^2} = \frac{8}{m}$$

$m^2-4m+4 = 8m \Rightarrow m^2-12m+4=0$

$\therefore$ $m_1+m_2 = 12$ and $m_1 m_2 = 4$

now $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2} = \frac{m_1^3+m_2^3}{(m_1m_2)^2}$

$$= \frac{(m_1+m_2)^3 - 3m_1m_2(m_1+m_2)}{(m_1m_2)^2}$$

$$= \frac{12^3-12.12}{16} = \frac{12^2.11}{16} = 99$$

Newton's Theorem

If α, β are roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then for $n > 2$, $n \in N$, we have $aS_n + bS_{n-1} + cS_{n-2} = 0$

Proof :

$a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2})$

$= \alpha^{n-2}(a\alpha^2 + b\alpha + c) + \beta^{n-2}(a\beta^2 + b\beta + c)$

$= 0$ since α, β satisfy the equation.

➤ **Example 4.** If α, β are roots of $x^2 - 3x + 1 = 0$, find the value of $\alpha^4 + \beta^4$.

➤ ***Solution*** For the equation

$S_1 = \alpha + \beta = 3, \qquad \alpha\beta = 1.$

$S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2 \times 1 = 7$

For n = 3, Newton's theorem gives

$S_3 - 3S_2 + S_1 = 0$

$\Rightarrow \quad S_3 = 3S_2 - S_1 = 3 \times 7 - 3 = 18$

For n = 4,

$S_4 - 3S_3 + S_2 = 0$

$\Rightarrow \quad S_4 = 3S_3 - S_2 = 3 \times 18 - 7 = 47$

$\therefore \quad \alpha^4 + \beta^4 = 47$

Alternative :

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

$= ((\alpha + \beta)^2 - 2\alpha\beta)^2 - 2(\alpha\beta)^2$

$= (3^2 - 2 \times 1)^2 - 2(1)^2$

$= 7^2 - 2 = 47$

➤ **Example 5.** If α & β are the roots of the equation $x^2 - ax + b = 0$ and $v_n = \alpha^n + \beta^n$, show that $v_{n+1} = a v_n - b v_{n-1}$ and hence obtain the value of $\alpha^5 + \beta^5$.

➤ ***Solution*** $\alpha + \beta = a$; $\alpha\beta = b$; $v_n = \alpha^n + \beta^n$

$v_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta^n - \beta\alpha^n$

$= a v_n - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

Now $\alpha^5 + \beta^5 = v_5 = a v_4 - b v_3 = a(a v_3 - b v_2) - b v_3$

$= (a^2 - b) v_3 - ab v_2$

$= (a^2 - b)[a v_2 - b v_1] - ab v_2$

$= (a(a^2 - b) - ab) v_2 - b(a^2 - b) v_1$

$= (a^3 - 2ab)(a^2 - 2b) - ab(a^2 - b)$

$= a^5 - 2a^3b - 2a^3b + 4ab^2 - a^3b + ab^2$

$= a^5 - 5a^3b + 5ab^2$

$\Rightarrow \quad v_{n+1} = a v_n - b v_{n-1}$

2.7 POLYNOMIAL FUNCTION OF ROOT

Let α be a root of the equation $ax^2 + bx + c = 0$. Let g(x) be a polynomial in x, then $g(\alpha)$ can be reduced to a linear expression in α using $a\alpha^2 + b\alpha + c = 0$.

$g(x) = (ax^2 + bx + c).h(x) + px + q$

$\therefore \quad g(\alpha) = p\alpha + q$

➤ **Example 1.** When $x = \dfrac{3 + 5\sqrt{-1}}{2}$, find the value of $2x^3 + 2x^2 - 7x + 70$ and show that it will be unaltered if $\dfrac{3 - 5\sqrt{-1}}{2}$ be substituted for x.

➤ ***Solution*** Let us form a quadratic equation whose roots are $\dfrac{3 \pm 5\sqrt{-1}}{2}$;

the sum of the roots = 3 ;

the product of the roots = $\dfrac{17}{2}$;

hence the equation is $2x^2 - 6x + 17 = 0$;

$\therefore$ $2x^2 - 6x + 17$ is a quadratic expression which vanishes for either of the values $\dfrac{3 \pm 5\sqrt{-1}}{2}$.

Now $2x^3 + 2x^2 - 7x + 70$

$= x(2x^2 - 6x + 17) + 4(2x^2 - 6x + 17) + 2$

$= x \times 0 + 4 \times 0 + 2 = 2$;

which is the numerical value of the expression in each of the supposed cases.

Practice Problems

1. If α and β are the roots of $x^2 + 3x - 2(x + 7) = 0$, compute $\alpha^3 + \beta^3$, $\alpha/\beta + \beta/\alpha$ and $\alpha^2\beta + \alpha\beta^2$.
2. If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.
3. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2 + 2(m + n)x + m^2 + n^2 = 0$
4. If β, β are roots of the equation $x^2 - px + q = 0$, find the value of
   (i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$
   (ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$
5. If α, β are roots of $ax^2 + bx + c = 0$,then find the value of $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$
6. Let α, β be the roots of the equation $x^2 + ax - \dfrac{1}{2a^2} = 0$, being a real parameter, prove that $\alpha^4 + \beta^4 \geq 2 + \sqrt{2}$.
7. If $x = -5 + 2\sqrt{-4}$ then find the value of the expression $x^4 + 9x^3 + 35x^2 - x + 4$.
8. If α, β are the roots of equation, $x^2 - 2x + 3 = 0$, find the equation whose roots are $\alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $\beta^3 - \beta^2 + \beta + 5$
9. If $x^2 - 3x + 4 = 0$, prove that $x^4 = 3x - 20$. If the roots of the equation $x^2 - 3x + 4 = 0$ are α and β, construct the equation whose roots are α^4 and β^4.

2.8 NATURE OF ROOTS

Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ implies whether the roots are real or imaginary. By analysing the expression $b^2 - 4ac$, called the discriminant D, we get an idea about the nature of roots.
Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ then

(i) $D > 0 \Leftrightarrow$ roots are real and distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real and coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary. If $p + iq$ is one root of a quadratic equation with real coefficients, then the other must be its conjugate $p - iq$ and vice versa . $(p, q \in R$ & $i = \sqrt{-1})$.

STUDY TIP Difference between Root and Solution:
A root of a polynomial equation may be real or imaginary while a solution has to be real. A quadratic equation having two distinct real roots is said to have two solutions. If it has two equal real roots then it has one solution. If it has two imaginary roots then we say that the equation has no solution.

CAUTION By complex numbers, we mean numbers which can be written in the form $a + i b$ where a, b are real numbers and $i = \sqrt{-1}$. This includes both real and imaginary numbers. Hence by saying that a quadratic equation has complex roots, it does not imply that the roots are imaginary.

Factorization

If the equation $ax^2 + bx + c = 0$ has real roots α and β, then we have the factorization

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \qquad ...(i)$$

Thus the quadratic polynomial $ax^2 + bx + c$ can be written as a product of linear factors, each linear factor having only real coefficients. If the given equation has no real roots, then a factorization of the form (i) with α and β real is impossible. Nevertheless, we can find a factorization of the form (i) with imaginary α and β . If the quadratic expression $ax^2 + bx + c$ is such that it has no factorization of the form (i) with real α and β, then we say the polynomial $ax^2 + bx + c$ is irreducible over R.

Since the given equation has real roots if and only if the discriminant is non-negative, we conclude that $ax^2 + bx + c$ is irreducible over R is and only if the discriminant $D = b^2 - 4ac$ is negative. A quadratic expression will be a perfect square of a linear expression if the discriminant of its corresponding equation is zero.

Note

1. If D_1 and D_2 are discriminant of two quadratic equations and $D_1 + D_2 \geq 0$ then atleast one of D_1 & $D_2 \geq 0$
   $\Rightarrow$ Atleast one of the equations has real roots.
2. $D_1 + D_2 < 0$
   $\Rightarrow$ At least one of D_1 and $D_2 < 0$.
   $\Rightarrow$ At least one of the equations has imaginary root.
3. If D_1 is the discriminant of the equation $a_1x^2 + b_1x + c_1 = 0$ and D_2 is of $a_2x^2 + b_2x + c_2 = 0$ and $(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x\, c_2) = 0$ ...(1)
   (i) $D_1 D_2 < 0$
   $\Rightarrow D_1 > 0$ & $D_2 < 0$ or $D_1 < 0$ & $D_2 > 0$
   Then equation (1) has two real roots.
   (ii) $D_1 D_2 > 0$
   Case I : $D_1 > 0$ & $D_2 > 0$
   Then equation (1) has four real roots.
   Case II : $D_1 < 0$ & $D_2 < 0$
   Then equation (1) has no real roots.
   (iii) $D_1 D_2 = 0$
   Case I : $D_1 > 0$ & $D_2 = 0$ or $D_1 = 0$ & $D_2 > 0$
   Then equation (1) has two equal real roots & two distinct roots.
   Case II : $D_1 < 0$ & $D_2 = 0$ or $D_1 = 0$ & $D_2 < 0$
   Then equation (1) has two equal real roots.
   Case III : $D_1 = 0$ & $D_2 = 0$
   Then equation (1) has two real roots each repeated twice.
4. If sum of the coefficients of equation $ax^2 + bx + c = 0$ vanishes i.e. $a + b + c = 0$, then $x = 1$ is a root of the equation. Similarly if $4a + 4b + c = 0$ then $x = 2$ is a root of the equation $ax^2 + bx + c = 0$.

➤ **Example 1.** If the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find k.

➤ ***Solution*** The condition for equal roots gives
$(k + 2)^2 = 9k$,
$k^2 - 5k + 4 = 0$,
$(k - 4)(k - 1) = 0$;
$k = 4$ or 1.

➤ **Example 2.** Find all values of the parameter a for which the quadratic equation
$(a+1)x^2 + 2(a + 1)x + a - 2 = 0)$ has
(a) two distinct roots, (b) no real roots,
(c) two equal roots.

➤ ***Solution*** For the quadratic equation $a \neq -1$ and the discriminant $D = 4(a+1)^2 - 4(a + 1)(a - 2)$
$= 4(a + 1)(a + 1 - a + 2) = 12(a + 1)$.
If $a > -1$, then $D > 0$and this equation has two distinct roots.
If $a < -1$ then $D < 0$, and this equation has no real roots.
This equation cannot have two equal roots since $D = 0$ only for $a = -1$, and this is not acceptable.

➤ **Example 3.** If the equation $ax^2 + 2bx + c = 0$ has real roots, a, b, c being real numbers and if m and n are real numbers such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

➤ ***Solution*** Since roots of the equation
$ax^2 + 2bx + c = 0$ are real
$\therefore\ (2b)^2 - 4ac \geq 0$
$\therefore\ b^2 - ac \geq 0$ ...(1)
and discriminant of $ax^2 + 2mbx + nc = 0$
is $D = (2mb)^2 - 4anc$
$D = 4m^2b^2 - 4anc$ ...(2)
from (1) $b^2 \geq ac$ ...(3)
and given $m^2 > n$ ...(4)
$\therefore\ b^2 m^2 \geq anc$
$\Rightarrow\ 4b^2m^2 - 4\,anc \geq 0$
$\Rightarrow\ D \geq 0$ {from (2)}
Hence roots of equation $ax^2 + 2mbx + nc = 0$ are real.

➤ **Example 4.** Show that the expression $x^2 + 2(a+b+c)x + 3(bc+ca+ab)$ will be a perfect square if $a = b = c$.

➤ ***Solution*** Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.

i.e. $4(a+b+c)^2 - 4.3\,(bc+ca+ab) = 0$

or $(a+b+c)^2 - 3\,(bc+ca+ab) = 0$

or $\frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2) = 0$

which is possible only when $a = b = c$.

➤ **Example 5.** If $p, q, r, s \in R$ and α & β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has two real roots.

➤ ***Solution*** $\alpha + \beta = -p$; $\alpha\beta = q$; $\alpha^4 + \beta^4 = r$; $\alpha^4\beta^4 = s$

Now $D = 16q^2 - 4(2q^2 - r)$

$= 8q^2 + 4r = 2(4q^2 + r)$

$= 2[2\alpha^2\beta^2 + \alpha^4 + \beta^4]$

$= 2[(\alpha^2 + \beta^2)^2 + 2\alpha^2\beta^2]$

$= 2[\{(\alpha + \beta)^2 - 2\alpha\beta\}^2 + 2\alpha^2\beta^2]$

$= 2(p^2 - 2q)^2 + 2q^2] > 0$

Hence the equation $x^2 - 4qx + 2q^2 - r = 0$ has two real roots.

➤ **Example 6.** If $a, b, c \in R$ and $x^2 + bx + c = 0$ has no real roots. Prove that equation

$x^2 + bx + c(x + a)(2x + b) = 0$ has real roots for every a.

➤ ***Solution*** The second equation is

$(1 + 2c)x^2 + [b + c(b + 2a)]x + abc = 0$

whose discriminant $D = [b + c(b + 2a)]^2 - 4abc(1 + 2c)$

Arranging D as a quadratic in a,

$D = 4a^2c^2 - 4abc^2 + b^2(c + 1)^2$ ...(1)

Now roots of the second equation will be real for all a if the last expression is non-negative for all a. The necessary and sufficient condition for this is that the discriminant D* of the last expression is non-positive.

Indeed $D^* = 16b^2c^4 - 16b^2c^2(c + 1)^2$

$= 16b^2c^2(c^2 - (c + 1)^2) = 16b^2c^2(-2c - 1)$ ...(2)

Now the original equation has no real roots

$\Rightarrow$ $b^2 - 4c < 0 \Rightarrow c > 0 \Rightarrow$ The expression in (2) is ≤ 0

$\Rightarrow$ The expression (1) is positive for all a.

Practice Problems

1. Find the values of a for which the roots of the equation $(2a - 5)x^2 - 2(a - 1)x + 3 = 0$ are equal.
2. Find the values of a for which the equation $a^2x^2 + 2(a + 1)x + 4 = 0$ has coincident roots.
3. If $a + b + c = 0$ and a, b, c are real, then prove that equation $(b - x)^2 - 4(a - x)(c - x) = 0$ has real roots and the roots will not be equal unless $a = b = c$.
4. For each of the following equations find the set of values of a for which the equation has real roots.
   (a) $ax^2 + 9x - 1 = 0$
   (b) $2x^2 + 4x + a = 0$
   (c) $x^2 + (a + 2)x + a = 0$
5. If $a \ne b$, prove that the roots of $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are imaginary.
6. If the roots of the equation $x^2 - 2cx + ab = 0$ are real and unequal, prove that the roots of $x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.
7. Prove that the equation $(bx + c)^2 - a^2x^4 = 0$ has atleast two real roots for all real values of a, b, c.
8. If the equation $x^2 - 2px + q = 0$ has two equal roots, then prove that the equation $(1 + y)x^2 - 2(p + y)x + (q + y) = 0$ will have its roots real and distinct only when y is negative and p is not unity.
9. Find the nature of roots of $(b - x)^2 - 4(a - x)(c - x) = 0$ where a, b, c are distinct real numbers.
10. Suppose p and q are real numbers which do not take simultaneously the values $p = 0$, $q = 1$. Suppose the equation $(1 - q + p^2/2)x^2 + p(1 + q)x + q(q - 1) + p^2/2 = 0$ has two equal roots. Prove that $p^2 = 4q$.
11. If the roots of the equation $x^2 + bx + c = 0$ are real, show that the roots of the equation $x^2 + bx + c(x + a)(2x + b) = 0$ are again real for every real number a.
12. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real root.

2.9 RATIONAL ROOTS

Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$ then ;

(i) If $-\frac{b}{a} \in Q$, $\frac{c}{a} \in Q$ and $D = b^2 - 4ac$ is a perfect square of a rational number then both roots are rational.

(ii) If $-\frac{b}{a} \in Q$, $\frac{c}{a} \in Q$ and $D = b^2 - 4ac > 0$ is not a perfect square of a rational number then both roots are irrational.

If $p + \sqrt{q}$ is one root in this case, (where p and q are rational) then the other root must be its conjugate i.e. $p - \sqrt{q}$ and vice-versa.

(iii) If $-\frac{b}{a} \in I$, $\frac{c}{a} \in I$ and $\frac{b^2 - 4ac}{4a^2}$ is a perfect square of an integer, then both the roots are integers. In particular when a = 1, b, c ∈ I and $b^2 - 4ac$ is a perfect square then both roots are integers.

Proof :

Let the equation $ax^2 + bx + c = 0$ have two integral roots, say α and β.

Since $\alpha, \beta \in I, \alpha + \beta = -\frac{b}{a} \in I$

$$\alpha\beta = \frac{c}{a} \in I$$

Also $\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{1}{2}\left(-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4.\frac{c}{a}}\right)$$

Let as assume $\frac{b}{a} = p$, $\frac{c}{a} = q$, p, q ∈ I.

Then $\alpha, \beta = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right)$

For integral roots, $p^2 - 4q$ must be a perfect square.

$\sqrt{p^2 - 4q}$ is even or odd depending on p being even or odd. In both situations α, β come out to be integers since $-p \pm \sqrt{p^2 - 4q}$ is always even.

Hence if $-\frac{b}{a} \in I$, $\frac{c}{a} \in I$ and $\frac{b^2 - 4ac}{4a^2}$ is a perfect square of an integer, then both the roots are integers. In the special case when a = 1, b, c ∈ I and $b^2 - 4ac$ is a perfect square then both roots are integers.

STUDY TIP It should be remarked that both roots are rational or both irrational, if the coefficients are rational, and that both roots are real or both imaginary, if the coefficients are real. If the coefficients are real, then a rational root and an irrational root may occur together and if the coefficients are complex, then a real root and an imaginary root may occur together.

➤ **Example 1.** Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational, where p, q, r are rational numbers.

➤ ***Solution*** The roots will be rational provided the coefficients are rational (which is obvious) and

$D = (-2p)^2 - 4(p^2 - q^2 + 2qr - r^2) = 4(q^2 - 2qr + r^2)$ $= 4(q - r)^2$ is a perfect square of a rational number. Hence the roots are rational.

➤ **Example 2.** Form a quadratic equation with rational coefficients if one of its root is $\cot^2 18°$.

➤ ***Solution*** $\cot^2 18° = \frac{1 + \cos 36°}{1 - \cos 36°} = \frac{1 + \frac{\sqrt{5}+1}{4}}{1 - \frac{\sqrt{5}+1}{4}}$

$$= \frac{(5+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}$$

$$= \frac{20 + 5\sqrt{5} + 3\sqrt{5}}{9 - 5} = \frac{20 + 8\sqrt{5}}{4} = 5 + 2\sqrt{5}$$

Hence if $\alpha = 5+2\sqrt{5}$ then $\beta = 5-2\sqrt{5}$

$\therefore$ $\alpha + \beta = 10;\ \alpha\beta = 25 - 20 = 5$

$\therefore$ The quadratic equation is $x^2 - 10x + 5 = 0$.

➤ **Example 3.** If both the roots of the quadratic equation $x^2 - (2n + 18)x - n - 11 = 0$, $n \in I$, are rational, then find the value(s) of n.

➤ ***Solution*** Here the coefficients are integers, hence rational.

Now the discriminant of given equation must be a perfect square of a rational number.

i.e. $4[(n + 9)^2 + n + 11]$ must be perfect square

$\Rightarrow$ $n^2 + 19n + 92$ must be perfect square of a whole number since $n \in I$

$\Rightarrow$ $n^2 + 19n + 92 = m^2$ where $m \in W$

$$\Rightarrow \quad n = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$$

$\Rightarrow$ $4m^2 - 7$ is a perfect square of a whole number

$\Rightarrow$ $4m^2 - 7 = p^2$ where $p \in W$

$\Rightarrow$ $4m^2 - p^2 = 7$

$\Rightarrow$ $(2m + p)(2m - p) = 7$

$\Rightarrow$ either $2m + p = \pm 1,\ 2m - p = \pm 7$

or $2m + p = \pm 7,\ 2m - p = \pm 1$

But $m, p \in W$

either $2m + p = 1,\ 2m - p = 7$

or $2m + p = 7,\ 2m - p = 1$

$2p = -6$ (not acceptable as $p \in W$)

$2p = 6,\ 2m = 4$

$\Rightarrow$ $m = 2$

$\Rightarrow$ $n^2 + 19n + 92 = 4$

$\Rightarrow$ $(n + 8)(n + 11) = 0 \Rightarrow n = -8$ or -11

➤ **Example 4**. Find $a \in I$ so that both the roots of the equation $ax^2 + (3a + 1)x - 5 = 0$ are integers.

➤ ***Solution*** We write the equation as

$$x^2 + \left(3 + \frac{1}{a}\right)x - \frac{5}{a} = 0.$$

$3 + \frac{1}{a} \in I$ when $a = \pm 1$.

$\frac{5}{a} \in I$ when $a = \pm 1, \pm 5$.

We try $a = \pm 1$ to check whether D is a perfect square or not. With $a = 1$, D is a perfect square. Hence the value of a is 1.

➤ **Example 5.** Find a, if $ax^2 - 4x + 9 = 0$ has integral roots.

➤ ***Solution*** Let $a = \frac{1}{b}$, so that the given equation becomes $x^2 - 4bx + 9b = 0$.

This equation has integral roots if b is an integer and $16b^2 - 36b$ is a perfect square

Let $b(4b - 9) = k^2 \Rightarrow 4b^2 - 9b - k^2 = 0$

$$\Rightarrow \left(2b - \frac{9}{4}\right)^2 - k^2 = \frac{81}{16} \Rightarrow (8b - 9)^2 - 16k^2 = 81$$

$\Rightarrow$ $(8b - 9 - 4k)(8b - 9 + 4k) = 81 = 3 \times 27$.

Since b and k are integers, $8b - 9 - 4k = 3$ and $8b - 9 + 4k = 27$

$\Rightarrow$ $16b - 18\ \ 30 \Rightarrow b = 3 \Rightarrow a = \frac{1}{3}$.

For any other factorization of 81, b will not be an integer.

➤ **Example 6.** Find all the integral values of a for which the quadratic expression $(x - a)(x - 10) + 1$ can be factored as a product of two factors $(x + \alpha)(x + \beta)$ where $\alpha, \beta \in I$.

➤ ***Solution*** We have $(x - a)(x - 10) + 1 = (x + \alpha)(x + \beta)$

Putting $x = -\alpha$ on both sides, we obtain

$(-\alpha - a)(-\alpha - 10) + 1 = 0$

$(\alpha + a)(\alpha + 10) = -1$

$\alpha + a$ and $\alpha + 10$ are integers

(since $a, \alpha \in I$)

$\therefore$ $\alpha + a = -1$ and $\alpha + 10 = 1$

or $\alpha + a = 1$ and $\alpha + 10 = -1$

(i) If $\alpha + 10 = -1$ then $\alpha = -9$ and $a = 8$. Similarly $\beta = -9$

Here $(x - 8)(x - 10) + 1 = (x - 9)^2$

(ii) If $a + 10 = -1$ then $\alpha = -11$ and $a = 12$. Similarly $\beta = 12$

Here $(x - 12)(x - 10) + 1 = (x - 11)^2$.

➤ **Example 7.** If $a \neq 1, -2$ and $a \in Q$, show that the roots of the equation $(a^2 + a - 2)\,x^2 + (2a^2 + a - 3)\,x + a^2 - 1 = 0$ are rational .

➤ ***Solution*** Notice that $x = -1$ satisfies this equation since

$$(a^2 + a - 2)\,(-1)^2 + (2a^2 + a - 3)(-1) + a^2 - 1 = 0.$$

Using product of roots, the other root is $\dfrac{1-a^2}{a^2+a-2}$ which is a rational number since $a \in Q$.
Hence both the roots are rational.

Practice Problems

1. Find all integral values of m for which the roots of the equation $mx^2 + (2m - 1)\,x + m - 2 = 0$ are rational.
2. If $a, b, c \in Q$ & $a + c \neq b$, then prove that the roots of the quadratic equation $(a + c - b)\,x^2 + 2cx + (b + c - a) = 0$ are rational.
3. If a, b c are rational numbers, show that the roots of the equation $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2\,(a^2 - b^2) = 0$ are rational.
4. Prove that the roots of the quadratic equation $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0\,(a, b, c \in Q)$ are rational
5. For what integral values of 'a' the equation $x^2 - x(1 - a) - (a + 2) = 0$ has integral roots.
6. Find the integral values of x, for which $x^2 + 7x + 13$ is a perfect square.
7. Suppose α, a, b are integers and $b \neq -1$. Show that if α satisfies the equation $x^2 + ax + b + 1 = 0$, then $a^2 + b^2$ is composite.

2.10 COMMON ROOTS OF QUADRATIC EQUATIONS

Two Common Roots

Let us find the condition that the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ may have two common roots.

The above equations have both roots common if they are identical. They are identical if and only if the coefficient of similar powers of x in the two equations are proportional.
Thus, $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have both roots common if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

STUDY TIP If it is known that an imaginary root is common in the above equations (where the coefficients are real) then the other root being its conjugate must also be common.
Similarly if it is known that an irrational root is common in the above equations (where the coefficients are rational) then the other root being its conjugate must also be common.
In these cases the condition for both common roots must be applied.

Atleast one Common Root

To find the condition that the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ may have atleast one common root.
Suppose these equations are both satistied by $x = \alpha$; then

$$a\alpha^2 + b\alpha + c = 0,$$
$$a'\alpha^2 + b'\alpha + c' = 0\,;$$

$\therefore$ by cross multiplication

$$\frac{\alpha^2}{(ca' - b'c)} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

To eliminate α, square the second of these equal ratios and equate it to the product of the other two; thus

$$\frac{\alpha^2}{(ca' - c'a)^2} = \frac{\alpha^2}{(bc' - b'c)} \cdot \frac{1}{ab' - a'b}\,;$$

$\therefore$ $(ca' - c'a)^2 = (bc' - b'c)\,(ab' - a'b)$,
which is the condition required.
Also note that the common root is

$$\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}\,.$$

STUDY TIP The above condition is also satisfied if the coefficients of the two equations are proportional. Hence this condition includes the situation of two common roots. By applying this condition we get atleast one root common.

Note

1. We can prove that this is also the condition that the two quadratic functions $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ may have a common linear factor.
2. If f(x) = 0 & g(x) = 0 are two polynomial equations having some common root(s) then those common root(s) is/are also the root(s) of h(x) = a f(x) + b g (x) = 0, but not all roots of h(x) are necessarily common roots.
3. To find the common root between the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ make the coefficient of x^2 same by multiplying the equations by a' and a respectively and substract the resulting equations.

➤ **Example 1.** If $a, b, c \in R$ and equations $ax^2+bx+c=0$ and $x^2 + 2x + 5 = 0$ have a common root, show that a : b : c = 1 : 2 : 5.

➤ ***Solution*** Given equations are : $x^2 + 2x + 5 = 0$ .... (i)

and $ax^2 + bx + c = 0$ .... (ii)

Clearly roots of equation (i) are imaginary. Since equations (i) and (ii) have a common root, the common root must be imaginary and hence both roots will be common.

Threfore equations (i) and (ii) are identical

$$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{5} \quad \Rightarrow \quad a : b : c = 1 : 2 : 5$$

➤ **Example 2.** If equations $ax^2 + bx + c = 0$ and $x^2 + 6x + 4 = 0$ have common root, show that a : b : c = 1 : 6 : 4.

➤ ***Solution*** Since roots of $x^2 + 6x + 4 = 0$ are irrational and equations $x^2 + 6x + 4 = 0$ & $ax^2 + bx + c = 0$ have a common root, both roots have to be common. Hence the equations have proportional coefficients.

$$\therefore \quad \frac{a}{1} = \frac{b}{6} = \frac{c}{4} \Rightarrow \quad a : b : c = 1 : 6 : 4$$

➤ **Example 3.** Find the value of p if the equation $3x^2 - 2x + p = 0$ and $6x^2 - 17x + 12 = 0$ have a common root.

➤ ***Solution*** Given equations are

$$3x^2 - 2x + p = 0 \quad ...(i)$$

and $$6x^2 - 17x + 12 = 0 \quad ...(ii)$$

Let α be the common root, then

$$3\alpha^2 - 2\alpha + p = 0 \quad ...(iii)$$

and $$6\alpha^2 - 17\alpha + 12 = 0 \quad ...(iv)$$

$$\therefore \quad \underset{(1)}{\frac{\alpha^2}{-24 + 17p}} = \underset{(2)}{\frac{\alpha}{6p - 36}} = \frac{1}{-51 + 12} = \underset{(3)}{\frac{1}{-39}}$$

from (1) and (2), $$\alpha = \frac{17p - 24}{6p - 36} \quad ...(A)$$

from (2) and (3)

$$\alpha = \frac{6p - 36}{-39} = -\frac{2p - 12}{13} \quad ...(B)$$

from (A) and (B), $$\frac{17p - 24}{6p - 36} = -\frac{2p - 12}{13}$$

or, $12p^2 - 144p + 432 = -221p + 312$

or, $12p^2 + 77p + 120 = 0$

or, $4p(3p + 8) + 15(3p + 8) = 0$

or, $(4p + 15)(3p + 8) = 0$

$$\therefore \quad p = -\frac{15}{4}, -\frac{8}{3}.$$

➤ **Example 4.** Find the values of 'k' so that the equations

$x^2 + kx + (k + 2) = 0$ and

$x^2 + (1 - k)x + 3 - k = 0$

have exactly one common root.

➤ ***Solution*** Using the condition of common root :

$(ca' - ac')^2 = (bc' - cb')(ab' - ba')$

We have $[(k + 2) . 1 - 1 . (3 - k)]^2 = [k(3 - k) - (k + 2)(1 - k)] . [(1 - k) . 1 - k . 1]$

$\Rightarrow \quad (2k - 1)^2 = 2(1 - 2k)(2k - 1)$

$$\Rightarrow \quad (2k - 1)^2 = 0 \Rightarrow \quad k = \frac{1}{2}$$

CAUTION

We donot immediately produce the answer as $k = \frac{1}{2}$. Here we need to check whether this value is associated with exactly one common root or both common roots.

We place the value of k is the given equations.

We get $x^2 + \frac{1}{x}x + \frac{5}{2} = 0$

$$x^2 + \frac{1}{x}x + \frac{5}{2} = 0$$

Since all the coefficients are equal with $k = \frac{1}{2}$, we get both common roots.

Hence, we donot have any value of k for which exactly one root is common.

➤ **Example 5.** If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

➤ ***Solution*** $\alpha^2 + b\alpha + c = 0$ ....(1)

$b\alpha^2 + c\alpha + 1 = 0$ ....(2)

$$\frac{\alpha^2}{b - c^2} = \frac{\alpha}{bc - 1} = \frac{1}{c - b^2}$$

$\Rightarrow \quad \alpha = \frac{b - c^2}{bc - 1}$ or $\alpha = \frac{bc - 1}{c - b^2}$

$\Rightarrow \quad (bc - 1)^2 = (b - c^2)(c - b^2)$

$\Rightarrow \quad b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow \quad (b + c + 1)(b^2 + c^2 + 1 - bc - c - b) = 0$

$\Rightarrow \quad b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

➤ **Example 6** Determine all possible value(s) of 'p' for which the equation $ax^2 - px + ab = 0$ and $x^2 - ax - bx + ab = 0$ may have a common root, given a, b are non zero real numbers.

➤ ***Solution*** $x^2 - (a + b)x + ab = 0$

or $(x - a)(x - b) = 0 \quad \Rightarrow \quad x = a$ or b

if $x = a$ is the root of other equation,

$a^3 - ap + ab = 0 \quad \Rightarrow \quad p = a^2 + b$

if $x = b$ is the root of the other equation, then $ab^2 - pb + ab = 0$; $\quad p = a(1 + b)$

Hence $p = a^2 + b$ or $a(1 + b)$

Practice Problems

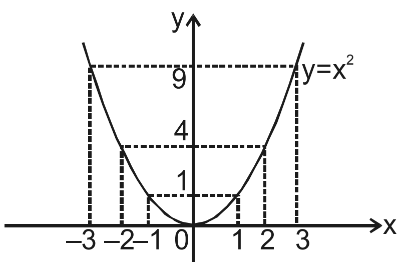
1. If the equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root, show that $2p^2 + 3q^2 - 5pq + 1 = 0$.
2. Prove that the equations $ax^2 + bx + c = 0$ and $2x^2 - 3x + 4 = 0$ cannot have a common root unless $6a = -4b = 3c$.
3. Prove that the equations $(q - r)x^2 + (r - p)x + p - q = 0$ and $(r - p)x^2 + (p - q)x + q - r = 0$ have a common root.
4. Let f(x) and g(x) be two quadratic polynomials all of whose coefficients are rational numbers. Suppose f(x) and g(x) have a common irrational root, then show that $g(x) = r f(x)$ for some rational number r.
5. If the equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, prove that their other roots satisfy the equation $x^2 - a(b + c)x + a^2bc = 0$.
6. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the first, then prove that $(q - b)^2 - bq(p - a)^2 = 0$
7. If the equation
   $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$
   has equal roots, and
   $4x^2 \sin^2\theta - 4\sin\theta . x + 1 = 0$ and the previous equation have a common root, find the value of θ.
8. Find λ, μ so that the equation $4x^2 - 8x + 3 = 0$, $x^2 + \lambda x - 1 = 0$ and $2x^2 + x + \mu = 0$ may have a common root for each pair of equations.

2.11 GRAPH OF QUADRATIC FUNCTION

Graph of $y = x^2$

Let us compile a table of its values :

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

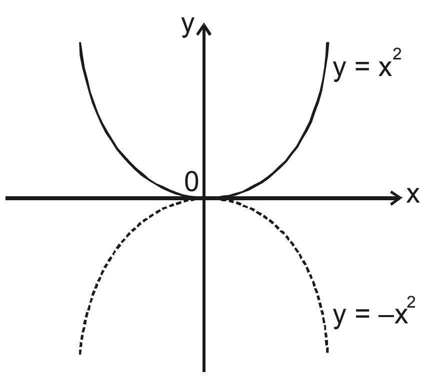


The graph of $y = x^2$ is called a parabola. Its graphs is symmetric about the y-axis. The point of intersection of the parabola with its axis of symmetry is called the vertex of the parabola. The vertex of the parabola $y = x^2$ is origin (0, 0).

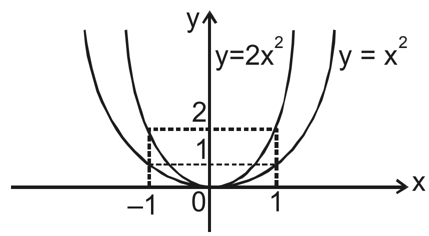
Also note that the function $y = x^2$ increases on the interval $(0, \infty)$ and decreases on the interval $(-\infty, 0)$.

Let us compare the function $y = -x^2$ and $y = x^2$. For the same value of x, the values of these functions are equal in magnitude but opposite in sign.

Consequently, the graph of the function $y = -x^2$ can be obtained by reflection of $y = x^2$ about the x-axis. The parabola $y = x^2$ opens upwards while $y = -x^2$ opens downward.

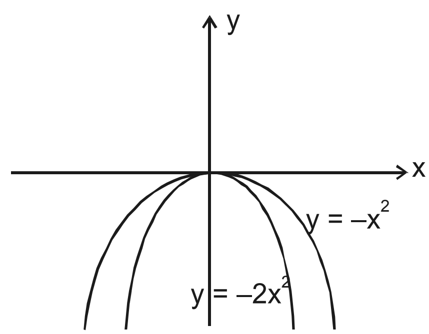


Let us now compare the functions $y = 2x^2$ and $y = x^2$. For the same value of x, the function $y = 2x^2$ is twice the value of the function $y = x^2$. Consequently, the graph of $y = 2x^2$ is obtained by stretching the graph of $y = x^2$, two times along the y–axis.

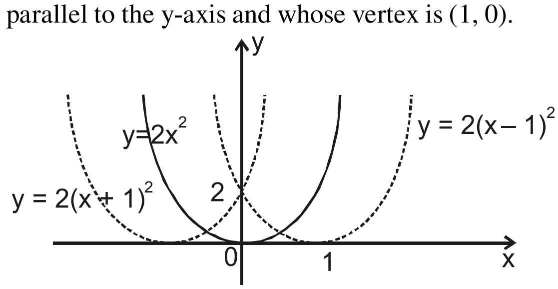


In general, the graph of $y = ax^2$ for $a > 0$ can be obtained by stretching the parabola $y = x^2$ a times along y axis (more precisely, by stretching for $a > 1$ and by compressing for $0 < a < 1$).

Similarly, the graph of $y = ax^2$ for $a < 0$ can be obtained by stretching the parabola $y = -x^2$, $|a|$ times along y-axis.



Now compare the functions $y = 2(x - 1)^2$ and $y = 2x^2$. The function y $2(x - 1)^2$ takes on the same value as the function $y = 2x^2$, but with the corresponding value of the argument increased by unity. Consequently, the graph of the function $y = 2(x - 1)^2$ can be obtained by shifting the parabola $y = 2x^2$ one unit to the right. As a result, we shall get the parabola $y = 2(x - 1)^2$ whose axis of symmetry is parallel to the y-axis and whose vertex is (1, 0).



Similarly the graph of $y = 2(x + 1)^2$ is obtained by shifting $y = 2x^2$ one unit to the left.

Now let us draw $y = 2x^2 + 1$. The graph of $y = 2x^2 + 1$ is a parabola with vertex (0, 1) and y-axis as its axis of symmetry. This parabola can be obtained by shifting the parabola $y = 2x^2$ along the y-axis by 1 units upward.

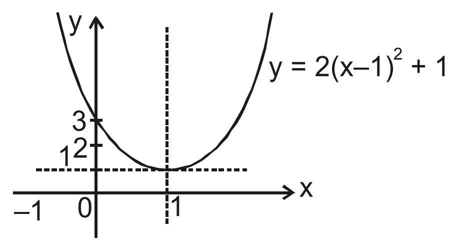
To contract the graph of $y = 2x^2 - 4x + 3$.

We rewrite the function $y = 2x^2 - 4x + 3$ by isolating a perfect square.

$$y = 2(x^2 - 2x + 1) + 1$$

$$\Rightarrow \quad y = 2(x - 1)^2 + 1.$$

The graph of $y = 2(x - 1)^2 + 1$ is a parabola with vertex (1, 1), whose axis is a straight line passing through its vertex parallel to the y-axis. Also note that the parabola opens upward since $a > 0$.



In general, $y = ax^2 + bx + c$

$$\Rightarrow \quad y = a.\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$$

$$\Rightarrow \quad y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

From the above considered particular cases if follows that the graph of $y = ax^2 + bx + c$ is a parabola with axis $x = -\frac{b}{2a}$ and vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, which opens upward if $a > 0$ and downward if $a < 0$.

The graph of a quadratic function can be constructed with the following technique :

(i) Plot the axis $x = -\frac{b}{2a}$ and the vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(ii) Find the real roots of the quadratic function, if any, and plot the corresponding points of the parabola on the x-axis.

(iii) Mark (0, c), the point of intersection of the parabola with the y-axis.

(iv) The parabola is drawn, with month opening upwards if $a > 0$ and downward if $a < 0$.

STUDY TIP It is easy to check that the x co-ordinate of the vertex is $\frac{\alpha + \beta}{2}$ where α, β are roots of the quadratic function. So, once we mark the roots (real) on the x-axis, we draw the axis of the parabola as a vertical line mid-way between the roots. If the roots are equal, the axis is drawn there itself.

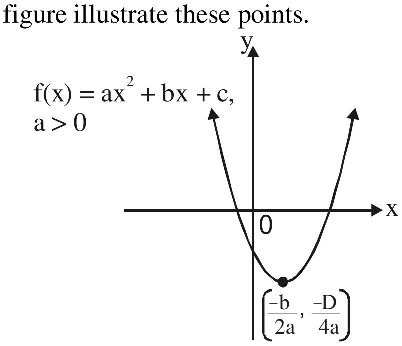
Extreme Point and Extreme Value of a Quadratic Function

For the quadratic function $P(x) = ax^2 + bx + c$,

a. if $a > 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the minimum point of the graph. The minimum value of the function is $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

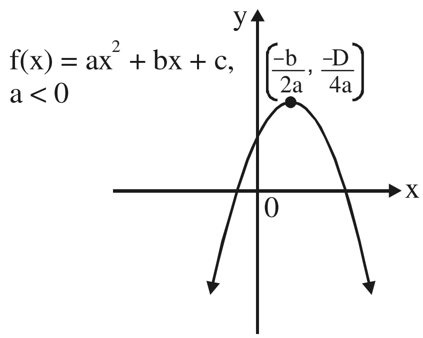
b. if $a < 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the maximum point of the graph. The maximum value of the function if $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

The figure illustrate these points.



$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the minimum point.

(a)



$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the maximum point.

(b)

➤ **Example 1**. Give the coordinates of the extreme point of the graph of each function, and the corresponding maximum or minimum value of the function.

(a) $f(x) = 2x^2 + 4x - 16$

(b) $f(x) = -x^2 - 6x - 8$

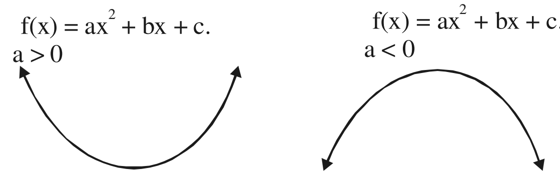
➤ ***Solution***

(a) The vertex of the graph of this function is (–1, 18). If opens upward since $a > 0$, so the vertex (–1, –18) is the minimum point and –18 is the minimum value of the function.

(b) The vertex (–3, 1) is the maximum point and $f(-3) = 1$ is the maximum value of the function.

End Behaviour of the Graph of a Quadratic Function

We know that if the value of a is positive for the quadratic function $f(x) = ax^2 + bx + c$, the graph opens upward, and if a is negative, the graph opens downward. The sign of a determines the end behaviour of the graph. If $a > 0$, as x approaches $-\infty$ or ∞ (writing $x \to -\infty$ or $x \to \infty$), the value of f(x) approaches ∞ (written $f(x) \to \infty$). The other situations similar to this are summarized below.

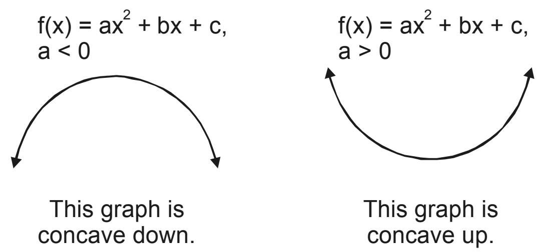


If $a > 0$,
as $x \to -\infty$, $f(x) \to \infty$;
as $x \to \infty$, $f(x) \to \infty$.

If $a < 0$,
as $x \to -\infty$, $f(x) \to -\infty$;
as $x \to \infty$, $f(x) \to -\infty$.

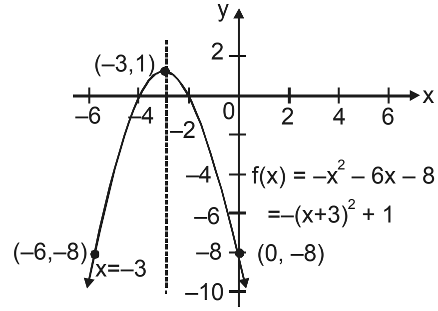
Concavity

Using the quadratic function graph as a illustration, we see that if $a > 0$, the graph is at all times opening upward. If water were to be poured from above, the graph would, in a sense, "hold water." We say that this graph is concave up for all values in its domain. On the other hand, if $a < 0$, the graph opens downward at all times, and it would similarly " dispel water" if it were poured from above. In this case, the graph is concave down for all values in its domain.



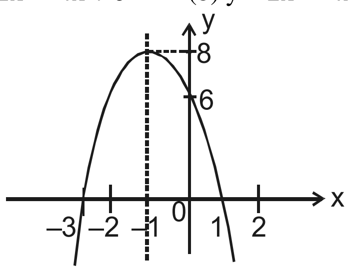
Note A formal discussion of concavity requires concepts beyond the scope of this text. It is studied more rigorously in calculus.

The graph of $f(x) = -x^2 - 6x - 8 = -(x + 3)^2 + 1$ is shown in the Figure

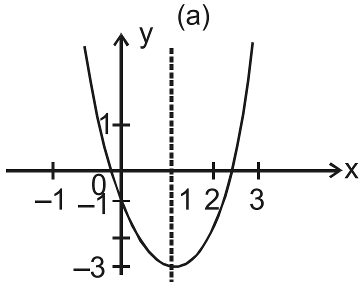


The parabola is the graph of $y = x^2$, translated 3 units to the left and 1 unit upward. It opens downward because of the negative sign before $(x + 3)^2$. The line $x = -3$ is its axis of symmetry, since if it were folded along this line, the two halves would coincide. The vertex, $(-3, 1)$, is the highest point on the graph. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 1]$. The function increases on the interval $(-\infty, -3]$ and decreases on $[-3, \infty)$. Since $f(0) = -8$, the y-intercept is -8, and since $f(-4) = f(-2) = 0$, the x-intercepts are -4 and -2.

➤ **Example 2.** Construct the graph of
(a) $y = -2x^2 - 4x + 6$ (b) $y = 2x^2 - 4x - 1$



(a)



(b)

➤ ***Solution***

(a) The vertex is $(-1, 8)$ using the formula $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. The axis of symmetry is $x = -1$. $y = -2x^2 - 4x + 6 = -2(x - 1)(x + 3)$. Hence the roots are $x = 1, -3$. Observe that $\frac{1 + (-3)}{2} = -1$ is the abscissa of the vertex. The graph has an y intercept of 6 units. Since $a = -2 < 0$, the parabola is drawn opening downward.

(b) We can transform $y = 2x^2 - 4x - 1$ as $y = 2(x - 1)^2 - 3$. Hence the vertex is $(1, -3)$.

The roots are $1 \pm \sqrt{\frac{3}{2}}$.

STUDY TIP **Graph of $f(x) = a(x - h)^2 + k$**

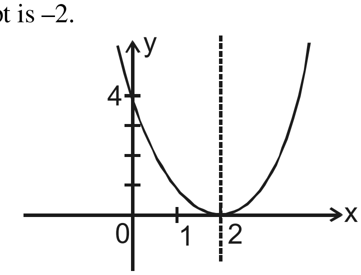
The graph of $f(x) = a(x - h)^2 + k$, $a \neq 0$,
(a) is a parabola with vertex (h, k), and the vertical line $x = h$ as the axis of symmetry;
(b) opens upward if $a > 0$ and downward if $a < 0$;
(c) is wider than $y = x^2$ if $0 < |a| < 1$ and narrower than $y = x^2$ if $|a| > 1$

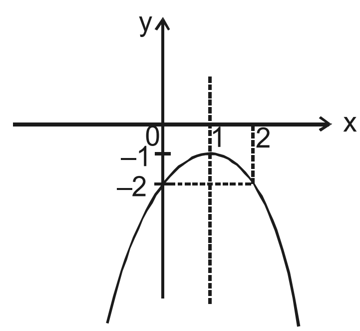
➤ **Example 3.** Construct the graphs of
(a) $y = x^2 - 4x + 4$ (b) $y = -x^2 + 2x - 2$

➤ ***Solution***

(a) The vertex is $(2, 0)$. Notice that the graph will touch the x-axis at $(2, 0)$ where $x = 2$ appears as a repeated root. The graph cuts the y-axis at $(0, 4)$.

(b) The vertex is $(1, -1)$. Since there are no real roots, the graph does not intersect the x-axis. The y intercept is -2.





Signs of a, b, c and D

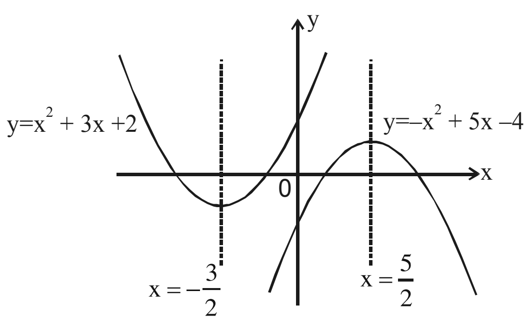
Consider the quadratic function $y = ax^2 + bx + c$.

1. The sign of 'a' is associated with the concavity of the parabola. If $a > 0$, then the parabola is

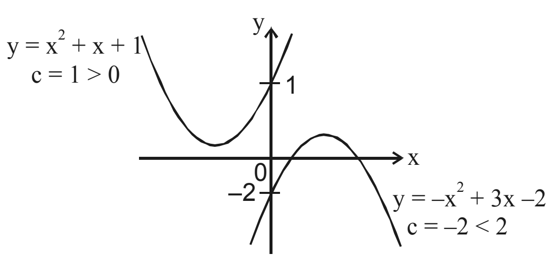
concave up and it is drawn with mouth opening upwards. If $a < 0$, then the parabola is concave down and it opens downward.

2. The sign of b is associated with the position of the axis of the parabola. Since the axis has the equation

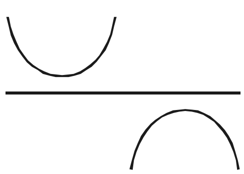
$x = -\dfrac{b}{2a}$, the axis is drawn to the left of y-axis if a & b are of same sign and it is drawn to the right of the y-axis if a & b are of opposite signs.



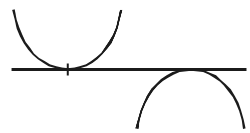
3. The sign of c is associated with the y-intercept of the parabola. If $c > 0$ then the parabola cuts the positive y-axis. If $c < 0$ then the parabola cuts the negative y-axis.



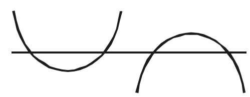
4. The sign of D is associated with the position of the parabola with respect to the x-axis.



D < 0

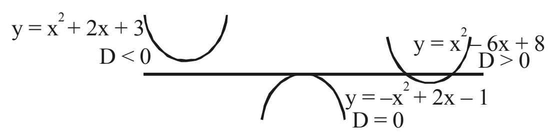


D = 0

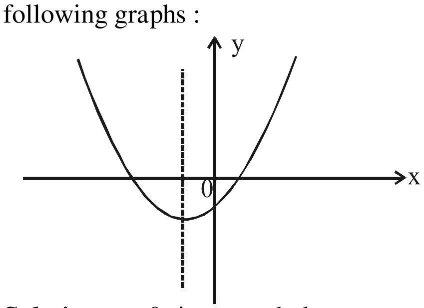


D > 0

The y-coordinate of the vertex is $-\dfrac{D}{4a}$. If $a > 0$, the vertex lies above, on or below the x-axis according as D is negative, zero or positive respectively. The number of times the parabola intersects the x-axis is equal to number of distinct real roots of the corresponding equation.



➤ **Example 4.** Predict the signs of a, b, c and D in the following graphs :



➤ ***Solution*** $a > 0$ since parabola opens upward.

$b > 0$ since $-\dfrac{b}{2a} < 0$ and $a > 0$.

$c < 0$ since y intercept is negative

$D > 0$ two distinct real roots

Roots using sign change of function

If the graph of a continuous function f(x) crosses the x-axis between β_1 and β_2 then f(x) changes sign between β_1 and β_2 i.e. $f(\beta_1)$ and $f(\beta_2)$ have opposite signs.

Hence, if $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, then $f(x) = 0$ has atleast one real root between β_1 and β_2. In case of a quadratic equation $f(x) = 0$ has exactly one real root between β_1 and β_2 in such situation. Also if $f(\beta_1)$ and $f(\beta_2)$ have same sign then $f(x) = 0$ may or maynot have real root between β_1 and β_2.

If a quadratic expression in x can be made to change its sign by giving real values to x, then the roots of the corresponding equation must be real.

Consider, for example, the expression

$a^2(x-\beta)(x-\gamma) + b^2(x-\gamma)(x-\alpha) + c^2(x-\alpha)(x-\beta)$, where the quantities are all real, and α, β, γ are supposed to be in order of magnitude. The expression is clearly positive if $x = \alpha$, and is negative if $x = \beta$. Hence the expression can be made to change its sign, and therefore the roots of the equation

$a^2(x-\beta)(x-\gamma) + b^2(x-\gamma)(x-\alpha) + c^2(x-\alpha)(x-\beta) = 0$

are real for all real values of a, b, c α, β, γ.

➤ **Example 5.** The coefficients of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, satisfy the inequality

$(a + b + c)(4a - 2b + c) < 0$. Prove that this equation has 2 distinct real solutions.

➤ ***Solution*** $ax^2 + bx + c = 0$ $\quad (a \neq 0)$

given $(a + b + c)(4a - 2b + c) < 0$

$f(1) = a + b + c \qquad f(-2) = 4a - 2b + c$

$f(1) \cdot f(-2) < 0$

since f (1) and f (–2) have opposite signs

$\therefore$ exactly one root lies between – 2 and 1

$\therefore$ this quadratic equation has 2 distinct real roots.

➤ **Example 6.** If $c < 0$ and the equation $ax^2 + bx + c = 0$, does not have any real root then prove that

(i) $a - b + c < 0$ $\qquad$ (ii) $9a + 3b + c < 0$.

➤ ***Solution*** Let $f(x) = ax^2 + bx + c$

$c = f(0) < 0$.

Since f(x) has no real root the graph of f(x) must open downward and should not cross the x-axis .

$\Rightarrow$ $f(x) = ax^2 + bx + c < 0$ for all $x \in R$

and hence $f(-1) = a - b + c < 0$

and $f(3) = 9a + 3b + c < 0$.

➤ **Example 7.** For what values of p, the number 6 lies between the roots of the equation $x^2 + 2(p-3)x + 9 = 0$

➤ ***Solution*** Since coefficient of x^2 is positive the graph of $f(x) = x^2 + 2(p-3)x + 9$ opens upward. If we have $f(6) < 0$ then one root will be less than 6 and other greater than 6.

$\Rightarrow$ $36 + 12(p-3) + 9 < 0$

$\Rightarrow$ $12p + 9 < 0$

$\Rightarrow$ $p < -3/4$

$p \in (-\infty, -3/4)$

➤ **Example 8.** If $a < b < c < d$, then prove that the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct.

➤ ***Solution*** Consider $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

$f(a) = 2(a-b)(a-d) > 0$

$f(b) = (b-a)(b-c) < 0$

$f(c) = 2(c-b)(c-d) < 0$

$f(d) = (d-a)(d-c) > 0$

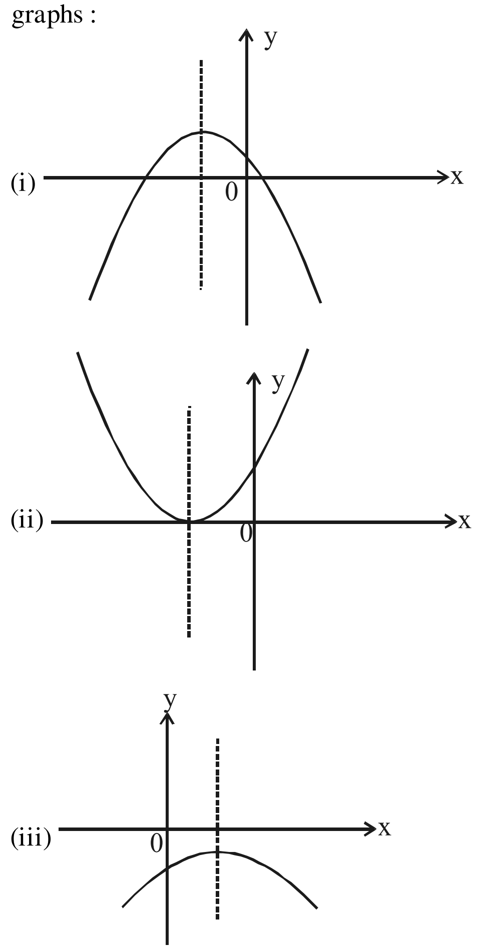
We see that f(x) changes sign between a and b and again between c and d. Hence $f(x) = 0$ has one real between a and b and other between c and d.

Practice Problems

1. Draw the graph of the following functions
   (a) $y = x^2 + 5x + 6$
   (b) $y = 4x^2 + 4x + 1$
   (c) $y = x^2 + x + 1$
2. (a) $y = 3x - x^2 - 2$
   (b) $y = 2x - x^2 - 1$
   (c) $y = x - x^2 - 1$
3. For what values of p does the vertex of the parabola $y = x^2 + 2px + 13$ lie at a distance of 5 units from the origin ?
4. Determine the intervals of the decrease of the function:
   (a) $y = x^2 - 3x + 1$ $\quad$ (b) $y = -x^2 - 4x + 8$
5. Determine the intervals of the increase of the function:
   (a) $y = \frac{1}{3}x^2 + x - \sqrt{2}$ $\quad$ (b) $y = -2x^2 + 8x - 3$
6. At what values of a does the function $f(x) = -x^2 + (a-1)x + 2$ increases on the interval (1, 2) ?

7. Find the values of k for which the curve $y = x^2 + kx + 4$ touches the x axis.

8. Predict the signs of a, b, c and D in the following graphs :



9. If $ax^2 + bx + c = 0$ has no real roots and $a + b + c < 0$, then find the sign of the number c.

10. Prove that both the roots of the equation $(x-4)(x-5) + (x-2)(x-5) + (x-2)(x-4) = 0$ are always real.

11. If a, b, c $\in$ R and $a - b + 2c = 0$ then prove that the roots of the equation $ax^2 + bx + c = 0$ are real and distinct.

12. Given $f(x) = ax^2 + bx + c$, $f(-1) > -4$, $f(1) < 0$ and $f(3) > 5$, find the sign of 'a'.

2.12 QUADRATIC INEQUALITY

A quadratic inequality is any inequality that can be put in one of the forms

$$2x^2 + bx + c < 0 \qquad ax^2 + bx + c > 0$$
$$ax^2 + bx + c \le 0 \qquad ax^2 + bx + c \ge 0$$

where a, b and c are real numbers and $a \ne 0$.
Consider, for example $\quad x^2 - 7x + 10 > 0$.
Because we are interested in the inequality $x^2 - 7x + 10$ where $y > 0$, focus on the portion of the graph of $y = x^2 - 7x + 10$ where y is positive. This occurs where the graph lies above the x-axis, namely, when $x < 2$ or $x > 5$. Thus, the solution set to $x^2 - 7x + 10 > 0$ is $(-\infty, 2) \cup (5, \infty)$
The solution set for $x^2 - 7x + 10 < 0$, occurs when the graph lies below the x-axis, between x = 2 and x = 5. Thus (2, 5), is the solution set for $x^2 - 7x + 10 < 0$.

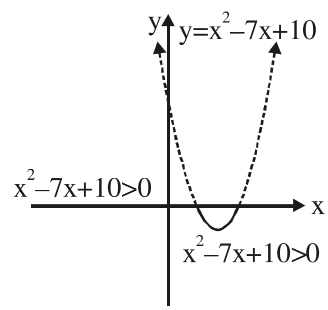


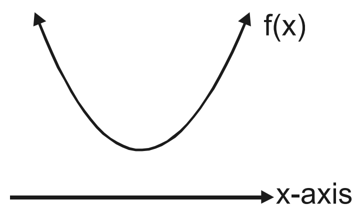
Figure
The doted portion of
$y = x^2 - 7x + 10$
represents
$x^2 - 7x + 10 > 0$

Solving Quadratic Inequalities

Any quadratic expression $f(x) \equiv ax^2 + bx + c$ has the same sign between its real roots and changes sign only when the graph of f(x) passes through any of its real roots. If f(x) has no real roots, then f(x) has the same sign for all real values and the sign of f(x) is determined by the sign of the coefficient a.
If f is a quadratic function, the solution sets of $f(x) = 0$, $f(x) < 0$, and $f(x) > 0$ can be found graphically based on the following summary.

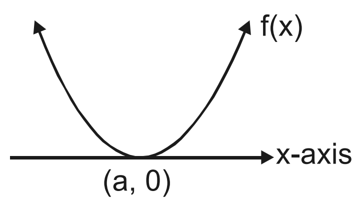
Different positions For Graphs of Quadratic Functions

I. $a > 0$ and $D < 0$



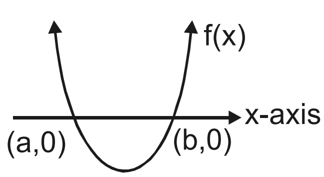
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \infty)$

II. $a > 0$ and $D = 0$



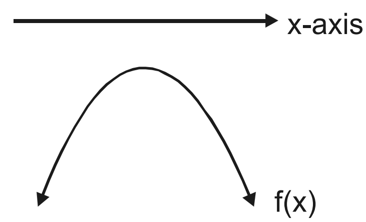
Question	Solution Set
$f(x) = 0$	$\{a\}$
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$

III. $a > 0$ and $D > 0$



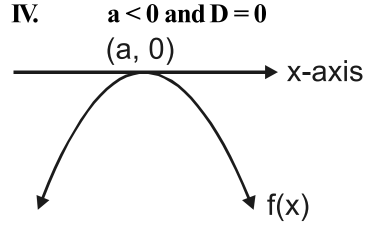
Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	(α, β)
$f(x) > 0$	$(-\infty, \alpha) \cup (\beta, \infty)$

IV. $a < 0$ and $D < 0$



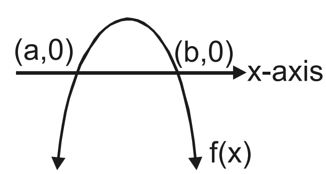
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	$(-\infty, \infty)$
$f(x) > 0$	ϕ

IV. $a < 0$ and $D = 0$



Question	Solution Set
$f(x) = 0$	$\{\alpha\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$
$f(x) > 0$	ϕ

VI. $a < 0$ and $D > 0$



Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\beta, \infty)$
$f(x) > 0$	(α, β)

Thus for real values of x the expression $ax^2 + bx + c$ has always the same sign as 'a' except for values of x which lie between the roots of the corresponding equation $ax^2 + bx + c = 0$.

STUDY TIP The values of 'x' satisfying the inequality,
$ax^2 + bx + c > 0$ $(a \neq 0)$ are :

(i) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $a < b$.

Then $a > 0 \Rightarrow \quad x \in (-\infty, a) \cup (b, \infty)$

$a < 0 \Rightarrow \quad x \in (a, b)$

(ii) If $D = 0$, i.e. roots are equal, i.e. $a = b$.

Then $a > 0 \Rightarrow \quad x \in (-\infty, a) \cup (a, \infty)$

$a < 0 \Rightarrow \quad x \in \phi$

(iii) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.

Then $a > 0 \Rightarrow \quad x \in R$

$a < 0 \Rightarrow \quad x \in \phi$

➤ **Example 1.** If $f(x) = x^2 + 2x + 2$, then solve the following inequalities :

(i) $f(x) \geq 0$ (ii) $f(x) \leq 0$

(iii) $f(x) > 0$ (iv) $f(x) < 0$

➤ ***Solution*** $f(x) = x^2 + 2x + 2$

Let us find $D = b^2 - 4ac = (2)^2 - 4 < 0$

$\Rightarrow \quad D < 0$

$\Rightarrow$ roots of the corresponding equation $(f(x) = 0)$ are imaginary.

Observe that the coefficient of $x^2 = a = 1 > 0$

As $a > 0$ and $D < 0$, we get :

$f(x) > 0 \ \forall \ x \in R$

(i) $f(x) \geq 0$ is true $\forall \ x \in R$

(ii) $f(x) \leq 0$ is true for no value of x i.e., $x \in \phi$

(iii) $f(x) > 0$ is true $\forall \ x \in R$

(iv) $f(x) < 0$ is true for no value of x i.e., $x \in \phi$

➤ **Example 2.** If $f(x) = x^2 + 4x + 4$, then solve the following inequalities :

(i) $f(x) \geq 0$ (ii) $f(x) \leq 0$

(iii) $f(x) > 0$ (iv) $f(x) < 0$

➤ ***Solution*** $f(x) = x^2 + 4x + 4 = (x + 2)^2$

Let us find D

$\Rightarrow \quad b^2 - 4ac = (4)^2 - 4(1)(4) = 0 \quad \Rightarrow \quad D = 0$

$\Rightarrow$ roots of the corresponding equation $(f(x) = 0)$ are real and equal.

Also observe that $a = 1 > 0$

As $D = 0$ and $a > 0$, we get :

$\Rightarrow \quad f(x) \geq 0 \ \forall \ x \in R$

(i) $f(x) \geq 0$ is true $\forall \ x \in R$

(ii) $f(x) \leq 0$ is true for $x \in \{-2\}$

(iii) $f(x) > 0$ is true $\forall \ x \in R - \{-2\}$

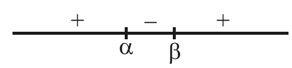
(iv) $f(x) < 0$ is true for no value of x i.e., $x \in \phi$

Method of Intervals

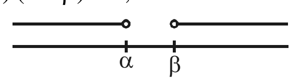
1. Express the inequality in the form $ax^2 + bx + c > 0$ or $ax^2 + bx + c < 0$
2. Solve the equation $ax^2 + bx + c = 0$. The real solutions are the critical numbers.
3. Locate these critical numbers on a number line, thereby dividing the number line into test intervals.
4. Choose one representative number for each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. The solution set is the union of all such test intervals.

Assuming that the discriminant $D > 0$

* Make the coefficient of x^2 positive.
* Factorize the expression and represent the left hand side of inequality in the form $(x - \alpha)(x - \beta)$.

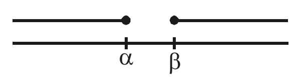


* If $(x - \alpha)(x - \beta) > 0$, then x lies outside α and β.



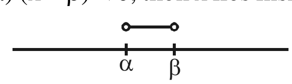
$\Rightarrow \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

* If $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside α and β.



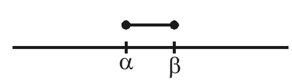
$\Rightarrow \ x \in (-\infty, \alpha] \cup [\beta, \infty)$

* If $(x - \alpha)(x - \beta) < 0$, then x lies inside α and β.



$\Rightarrow \ x \in (\alpha, \beta)$

* If $(x - \alpha)(x - \beta) \leq 0$, then x lies on and inside α and β.



$\Rightarrow \ x \in [\alpha, \beta]$

➤ **Example 3.** Solve $2x^2 + x < 15$

➤ ***Solution*** Write the inequality in standard form

$$2x^2 + x - 15 < 0$$

Solve the related quadratic equation

$2x^2 + x - 15 = 0 \quad \Rightarrow \quad x = 5/2, x = -3$

Factorize

$(2x - 5)(x + 3) < 0$

Use sign analysis of the product $(2x - 5)(x + 3)$

If $\quad x < -3, 2x - 5 < 0$ and $x - 3 < 0$

hence $\quad (2x - 5)(x + 3) > 0$

If $\quad -3 < x < \frac{5}{2}, (2x - 5) < 0$ and $x + 3 > 0$

hence $\quad (2x - 5)(x + 3) < 0$

If $\quad x > \frac{5}{2}, \quad (2x - 5) > 0$ and $(x + 3) > 0$

hence $\quad (2x - 5)(x + 3) > 0$

This indicates that $(2x - 5)(x + 3)$ is negative in the interval $\left(-3, \frac{5}{2}\right)$.

Another option is to use test points from each of the subintervals.

For the interval $(-\infty, -3)$, we substitute $x = -4$ into the inequality.

$(2.(-4)-5)(-4 + 3) = 13 > 0$

For $\left(-3, -\frac{5}{2}\right)$ we put $x = 0 \Rightarrow -15 < 0$

For $\left(\frac{5}{2}, \infty\right)$ we put $x = 3 \Rightarrow 6 > 0$

Conclusion : $\left(-3, \frac{5}{2}\right)$ is the solution of the inequality.

➤ **Example 4**. Solve the following quadratic inequality:

$x^2 - 2x - 3 < 0$

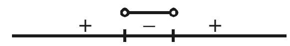
➤ ***Solution*** $x^2 - 2x - 3 < 0$

Let us find $D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 16 > 0$

$\Rightarrow \quad D > 0$

Now factorize

$x^2 - 3x + x - 3 < 0$



$\Rightarrow \quad (x - 3)(x + 1) < 0$

$\Rightarrow \quad x \in (-1, 3)$

➤ **Example 5.** For what values of k is the inequality

$x^2 - (k - 3)x - k + 6 > 0$ valid for all real x ?

➤ ***Solution*** The expression $x^2 - (k - 3)x - k + 6$ is positive for all x if coefficient of x^2 is positive, which is true and discriminant $D < 0$

$\Rightarrow \quad (k - 3)^2 - 4(-k + 6) < 0$

$\Rightarrow \quad k^2 - 2k - 15 < 0$

$\Rightarrow \quad (k + 3)(k - 5) < 0$

$k \in (-3, 5)$,

➤ **Example 6.** For what values of k, is the quadratic trinomial

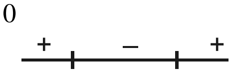
$(k - 2)x^2 + 8x + k + 4$ negative for all values of x ?

➤ ***Solution*** The quadratic expression is negative for all x if $k - 2 < 0$ and

$D = 8^2 - 4(k - 2)(k + 4) < 0$

$16 - (k^2 - 2k - 8) < 0$

$k^2 + 2k - 24 > 0$



$(k + 6)(k - 4) > 0$

$k < -6$ or $k > 4$.

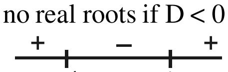
Taking intersection with $k < 2$, we get $k < -6$.

➤ **Example 7.** For what values of m does the equation

$mx^2 - (m + 1)x + 2m - 1 = 0$ possess no real roots.

➤ ***Solution*** The equation has no real roots if $D < 0$

$\Rightarrow \quad (m + 1)^2 - 4m(2m - 1) < 0$



$\Rightarrow \quad -7m^2 + 6m + 1 < 0$

$\Rightarrow \quad m \in \left(-\infty, -\frac{1}{7}\right) \cup (1, \infty)$

➤ **Example 8.** Find the values of k so that y = kx is secant to the curve $y = x^2 + k$.

➤ ***Solution*** put y = kx in $y = x^2 + k$

$kx = x^2 + k = 0$

$x^2 - kx + k = 0$ for line to be secant, D > 0

$k^2 - 4k > 0 \quad k(k-4) > 0$

hence $\quad k > 4$ or $k < 0$

$k \in (-\infty, 0) \cup (4, \infty)$

➤ **Example 9.** Show that (x – 1) (x – 3) (x – 4) (x – 6) + 10 is positive for all real values of x.

➤ ***Solution*** Taking the first and last factors together, and also the other two the given expression becomes

$(x^2 - 7x + 6)(x^2 - 7x + 12) + 10$

$= (x^2 - 7x)^2 + 18(x^2 - 7x) + 82$

Assuming this as a quadratic expression in $(x^2 - 7x)$,

$D = 18^2 - 4 \times 82 = 4(81 - 82) = -4 < 0$

which is clearly always positive for real values of c.

➤ **Example 10** Solve the inequality

$$ax^2 - 2x + 4 > 0. \qquad (1)$$

➤ ***Solution*** Equating to zero the coefficient of x^2 and the discriminant of the quadratic trinomial $ax^2 - 2x + 4$, we find the first singular value of the parameter a = 0 and the second singular value a = 1/4 (and if a > 1/4, then D < 0; and if a ≤ 1/4, then D ≥ 0).

Let us solve inequality (1) in each of the following four cases :

(1) a > 1/4; (2) 0 < a ≤ 1/4;

(3) a = 0; (4) a < 0.

(1) If a > 1/4, then the trinomial $ax^2 - 2x + 4$ has a negative discriminant and a positive leading coefficient. Hence, the trinomial is positive for any x, that is, the solution of inequality (1) in this cases is represented by the set of all real numbers.

(2) If 0 < a ≤ 1/4, then the trinomial $ax^2 - 2x + 4$ has the following roots :

$$x_{1,2} = \frac{1 \pm \sqrt{1-4a}}{a},$$

where $\quad \dfrac{1-\sqrt{1-4a}}{a} \le \dfrac{1+\sqrt{1-4a}}{a}$.

Hence, the solution of inequality (1) is represented by the following collection:

$$x < \frac{1-\sqrt{1-4a}}{a}\,;\; x > \frac{1+\sqrt{1-4a}}{a}.$$

(3) If a = 0, then inequality (1) takes the form: –2x + 4 >0, whence we get: x < 2.

(4) If a < 0, then we have

$$\frac{1+\sqrt{1-4a}}{a} < \frac{1-\sqrt{1-4a}}{a}.$$

Hence, in this case the solution of inequality (1) is represented by the following system:

$$\frac{1+\sqrt{1-4a}}{a} < x < \frac{1-\sqrt{1-4a}}{a}.$$

Answer :

(1) if a > 1/4, then $-\infty < x < +\infty$;

(2) if 0 < a ≤ 1/4, then $x < \dfrac{1-\sqrt{1-4a}}{a}$;

$x > \dfrac{1+\sqrt{1-4a}}{a}$;

(3) if a = 0, then x < 2;

(4) if a < 0, then $\dfrac{1+\sqrt{1-4a}}{a} < x < \dfrac{1-\sqrt{1-4a}}{a}$.

➤ **Example 11.** Find the values of a so that the function $f(x) = (a+1)x^2 - 3ax + 4a$ is negative for atleast one real x.

➤ ***Solution***

Case A :

If a + 1 < 0 then the graph of f(x) is a parabola which opens downward. Hence f(x) is definitely negative for some real x.

Case B :

If a + 1 > 0 then the graph of f(x) is a parabola which opens upward. Here f(x) is negative for atleast one real x if its graph crosses the x-axis, for which D > 0 is required.

$9a^2 - 4(a+1).4a > 0$

$\Rightarrow \quad 7a^2 + 16a < 0$

$\Rightarrow \quad -16/7 < a < 0$

Taking intersection we get – 1 < a < 0

Case C :
If $a + 1 = 0$ then the graph of $f(x) = 3x - 4$, is a straight line which goes below the x-axis for $x < 4/3$. Hence $f(x)$ is definitely negative for some real x.
So the final answer is $a < 0$

➤ **Example 12.** Prove the following inequality

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

(This is known as Cauchy-Schwartz Inequality)

➤ ***Solution*** Let $E = (a_1x + b_1)^2 + (a_2x + b_2)^2 + ... + (a_nx + b_n)^2$
We can see that the above sum is positive for every $x \in R$. Expanding, we have

$$E = (a_1^2 + a_2^2 + ... + a_n^2)x^2 + 2(a_1b_2 + a_2b_2 + ... + a_nb_n)x + (b_1^2 + b_2^2 + ... + b_n^2)$$

which will be positive for every $x \in R$ only if

coeff. of $x^2 +$ $a_1^2 + a_2^2 + ... + a_n^2 > 0$, which is true

and $D \le 0$

$$\Rightarrow \quad (a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

which is the desired result.

Practice Problems

1. Solve the inequalities :
(i) $x^2 - 3x - 4 \le 0$ (ii) $4x^2 + 4x + 1 \le 0$

2. Determine in each of the following problems the set of values of x for which the given inequality is true :
(A) $x^2 + 4x + 12 > 0$ (B) $x^2 - 4x + 3 < 0$
(C) $21 - 7(2x - 9) > 7x^2$
(D) $3x^2 - 5x - 8 \le 0$

3. Solve the inequalities :

(i) $\begin{cases} x^2 - 2x - 3 > 0 \\ x + 4 \ge 0 \end{cases}$

(ii) $\begin{cases} x^2 - 4 \ge 0 \\ x^2 - 2x - 8 \ge 0 \\ -x^2 + 5x - 4 \ge 0 \end{cases}$

(iii) $\dfrac{x^2 - 13x + 40}{\sqrt{19x - x^2 - 78}} \le 0$

4. Solve $x^2 + ax + a > 0$ for different real values of a.

5. Find the integral values of k for which the equations $(k - 12)x^2 + 2(k - 12)x + 2 = 0$ possess no real roots.

6. Determine the values of k for which the equation $\dfrac{x^2 + x + 2}{3x + 1} = k$ has both roots real.

7. For what values of a is the inequality $ax^2 + 2ax + 0.5 > 0$ valid throughout the entire number axis ?

8. For what integral k is the inequality $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ valid for any real x?

9. Find all values of a for which the inequality $(a - 1)x^2 - (a + 1)x + a + 1 > 0$ is satisfied for all real x.

10. Find all values of 'a' for which the inequality $(a + 4)x^2 - 2ax + 2a - 6 < 0$ is satisfied for all $x \in R$.

11. For what values of 'p' do the graphs of $y = 2px + 1$ and $y = (p - 6)x^2 - 2$ not intersect ?

12. Find the least integral value of k for which the equation $x^2 - 2(k + 2)x + 12 + k^2 = 0$ has two different real roots.

13. Find the values of the parameter 'a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} < 2$.

14. If $a + b + c = 0$ then prove that $4ax^2 + 3bx + 2c = 0$ must have real roots.

15. Given the inequality $ax + k^2 > 0$. For which values of a
(a) is the inequality valid for all values of x and k ?
(b) are there values of x and k for which the inequality is valid ?
(c) is there a value of x such that the inequality is valid for any value of k ?
(d) is there a value of k for which the inequality is valid for every value of x ?
(e) is there a value of k such that the inequality is valid for all values of x ?
(f) if there a value of x for which the inequality is valid for every value of k ?

2.13 RANGE OF QUADRATIC FUNCTION

Consider the quadratic function $f(x) = a x^2 + b x + c$

(i) Range when $x \in R$

If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \Rightarrow f(x) \in \left[-\infty, \frac{D}{4a}\right)$

Maximum or Minimum Value of
$y = ax^2 + bx + c$ occurs at $x = -(b/2a)$
according as $a < 0$ or $a > 0$.

➤ **Example 1.** Find maximum or minimum values of the following polynomials over $x \in R$.
(i) $f(x) = 4x^2 - 12x + 15$
(ii) $f(x) = -3x^2 + 5x - 4$

➤ ***Solution***

(i) $f(x) = 4x^2 - 12x + 15$

As $a = 4 > 0$

f(x) has minimum value at vertex

$$f_{min} = \frac{-D}{4a} \quad \text{at} \quad x = \frac{-b}{2a}$$

$D = (-12)^2 - 4 \times 4 \times 15 = 144 - 240 = -96$

$$\Rightarrow f_{min} = \frac{-(-96)}{4 \times 4} = \frac{96}{16} = 6 \text{ at } x = -\frac{-12}{2 \times 4} = \frac{3}{2}$$

$$\therefore f_{min} = 6 \text{ at } x = \frac{3}{2}$$

$f_{max} = \infty$

Alternative : Method of perfect square
Let us isolate a perfect square

$$f(x) = 4\left(x^2 - 3x + \frac{9}{4}\right) + 15 - 9 = 4\left(x - \frac{3}{2}\right)^2 - 6$$

$$f_{min} = -6 \text{ at } x = \frac{3}{2}$$

(ii) $f(x) = -3x^2 + 5x - 4$

As $a = -3 < 0$

f(x) has maximum value at vertex

$$f_{max} = \frac{-D}{4a} \text{ at } x = -\frac{b}{2a}$$

$D = (5)^2 - 4(-3)(-4) = 25 - 48 = -23$

$$f_{max} = -\frac{(-23)}{4(-3)} = -\frac{23}{12} \text{ at } x = \frac{-(5)}{2(-3)} = \frac{5}{6}$$

$$\therefore f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

$f_{min} = -\infty$

Alternative : Method of perfect square

$$f(x) = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) - 4 + \frac{25}{12}$$

$$= -3\left(x - \frac{5}{6}\right)^2 - \frac{23}{12}$$

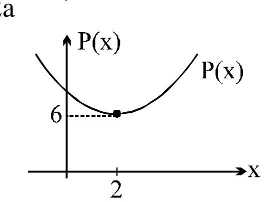
$$f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

➤ **Example 2.** Let $P(x) = ax^2 + bx + 8$ is a quadratic polynomial. If the minimum value of P(x) is 6 when x = 2, find the values of *a* and *b*.

➤ ***Solution*** $P(x) = ax^2 + bx + 8$ ....(1)

$P(2) = 4a + 2b + 8 = 6$ ....(2)

$$\therefore -\frac{b}{2a} = 2; \quad \therefore 4a = -b$$



from (2), we get

$-b + 2b = -2 \quad \Rightarrow \quad b = -2$

$\therefore \quad 4a = -(-2) \Rightarrow \quad a = 1/2$

➤ **Example 3.** If min $(x^2 + (a-b)x + (1-a-b)) >$ max $(-x^2 + (a+b)x - (1+a+b))$, prove that $a^2 + b^2 < 4$.

➤ ***Solution*** Let $f(x) = x^2 + (a-b)x + (1-a-b)$

$f(x)_{min} = -\dfrac{D_1}{4}$, where D_1 is the discriminant of f(x).

Let $g(x) = -x^2 + (a+b)x - (1+a+b)$.

$g(x)_{max} = \dfrac{-D_2}{-4}$ where D_2 is the discriminant of g(x).

$$\text{Thus } -\frac{(a-b)^2 - 4(1-a-b)}{4} > -\frac{(a+b)^2 - 4(1+a+b)}{-4}$$

$\Rightarrow \quad 4(1-a-b) - (a-b)^2 > (a+b)^2 > (a+b)^2 - 4(1+a+b)$

or $\quad 8 > 2(a^2 + b^2) \Rightarrow a^2 + b^2 < 4$.

➤ **Example 4.** Consider the quadratic polynomial $f(x) = x^2 - 4ax + 5a^2 - 6a$.

(a) Find the smallest positive integral value of 'a' for which f (x) is positive for every real x.

(b) Find the largest distance between the roots of the equation f (x) = 0.

➤ ***Solution***

(a) $D < 0$, $16a^2 - 4(5a^2 - 6a) < 0$, $4a^2 - 5a^2 + 6a < 0$, $a^2 - 6a > 0$,

$a(a-6) > 0 \Rightarrow a > 6$ or $a < 0$

$\therefore$ smallest +ve integer = 7

(b) $d = |\alpha - \beta|$, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 16a^2 - 4(5a^2 - 6a) = -4a^2 + 24a = -4(a^2 - 6a)$

$= -4[(a-3)^2 - 9] = 36 - 4(a-3)^2$

$\therefore \quad |\alpha - \beta|^2_{max.} = 36$ when $a = 3 \quad d_{max} = 6$

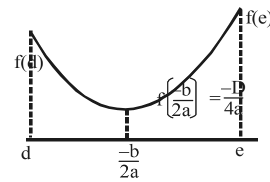
(ii) Range in restricted domain

Consider $\quad f(x) = ax^2 + bx + c, \qquad x \in [d, e]$.

Evaluate f(x) at the following points and choose the least and greatest values of f :

(i) f(d) $\qquad$ (ii) f(e)

(iii) If $x = \dfrac{-b}{2a} \in (d, e)$ then find $\left(-\dfrac{b}{2a}\right)$ or $-\dfrac{D}{4a}$



otherwise leave the vertex point since it does not lie in the restricted domain.

If the least value is m and the greatest value is M then the range is [m, M].

STUDY TIP If x belongs to a union of several intervals then the range is equal to the union of ranges found on each of these intervals.

➤ **Example 5.** Find the range of $f(x) = 2x^2 - 3x + 2$ in [0, 2].

➤ ***Solution*** $f(0) = 2$

$f(2) = 4$

$f\left(\dfrac{3}{4}\right) = \dfrac{7}{8}$ since $x = \dfrac{-b}{2a} = \dfrac{3}{4} \in [0, 2]$

$f\left(\dfrac{3}{4}\right)$ is the least value and f(2) is the greatest value. Then the range is $\left[\dfrac{7}{8}, 4\right]$.

Alternative : Method of perfect square

$$f(x) = 2\left(x - \frac{3}{4}\right)^2 + \frac{7}{8}, x \in [0, 2].$$

To obtain the least value $x = \dfrac{3}{4}$ is required to make the square zero. The greatest value is obtained using $x = 2$ since $\left|2 - \dfrac{3}{4}\right| > \left|0 - \dfrac{3}{4}\right|$.

Hence the range is [0, 2].

➤ **Example 6** Find the range of $y = 3 - 2\sin^2\theta - 6\sin\theta$.

➤ ***Solution*** Let $x = \sin\theta$, $x \in [-1, 1]$.

$f(x) = 3 - 2x^2 - 6x, x \in [-1, 1]$

$f(-1) = 3 - 2 + 6 = 7$

$f(1) = 3 - 2 - 6 = -5$

$$x = \frac{-b}{2a} = \frac{-6}{2 \times 2} = \frac{-3}{2} \notin [-1, 1].$$

Hence, we donot include the vertex. Range is $y \in [-5, 7]$.

Alternative :

$$f(x) = 3 - 2\left(x^2 + \frac{2.3}{2}x + \left(\frac{3}{2}\right)^2\right) + \frac{9}{2}$$

$$= \frac{15}{2} - 2\left(x + \frac{3}{2}\right)^2, x \in [-1, 1]$$

The least value is obtained when the square is made larger using $x = 1$.

The greatest value is obtained when the square is made smaller using $x = -1$.

Here we cannot make the square zero since $x = \frac{-3}{2}$ is not available.

Practice Problems

1. Find the maximum and minimum values of the functions :
   $f(x) = 2x^2 - 6x + 3$ where (i) $x \in R$ (ii) $x \in [2, \infty)$ (iii) $x \in [1, 5]$
2. Find the least and the greatest value of the functions in the indicated intervals :
   (a) $y = 3x^2 - x + 5$ on the interval [1 , 2]
   (b) $y = -4x^2 + 5x - 8$ on the interval [2 , 3]
   (c) $y = x^2 - 2x + 5$ on the interval [–1, 2]
   (d) $y = -x^2 + 6x - 1$ on the interval [0, 4]
3. Find the difference between the least and greatest values of $y = -2x^2 + 3x - 2$ for $x \in [0, 2]$.
4. If x be real find the maximum value of $3 - 6x - 8x^2$ and the corresponding value of x.
5. If equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then find the least value of $3a + b$.
6. Find the range of $y = 3^{x+1} - 2 \cdot 3^{2x} - 2$
7. Find the values of k so that the least value of the expression $x^2 + 2kx + k^2 + 3k$ for $x \in [0, 2]$ is 4.

2.14 RATIONAL FUNCTION

A rational function of x is defined as the ratio of two polynomials of x, say P(x) and Q(x) where $Q(x) \neq 0$ i.e. f(x) is a rational function of x if

$$f(x) = \frac{P(x)}{Q(x)}; Q(x) \neq 0,$$

Following are some examples of rational functions of x.

$$f(x) = \frac{2x + 1}{x^2 + x + 1} \quad ;$$

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 5x + 6} \quad ; \qquad x \neq 2, x \neq 3$$

$$f(x) = \frac{x^4 + 2x^3 + 3x + 1}{(x - 1)^2} \quad ; \qquad x \neq 1$$

➤ **Example 1.** If for any real x, we have

$$-1 \le \frac{x^2 + nx - 2}{x^2 - 3x + 4} \le 2,$$ then find the values of n.

➤ ***Solution*** The right hand inequality can be written

as $\frac{x^2 + nx - 2}{x^2 - 3x + 4} - 2 \le 0$

$$\Rightarrow \quad \frac{x^2 - (n + 6)x + 10}{x^2 - 3x + 4} \ge 0$$

$\Rightarrow \quad x^2 - (n + 6)x + 10 \ge 0$

[As $x^2 - 3x + 4 > 0 \; \forall \; x \in R$, since $D < 0$ and $a > 0$]

The above inequality will be true $\forall \; x \in R$ if its $D \le 0$

This gives $-\sqrt{40} - 6 \le n \le \sqrt{40} - 6$ ...(1)

The left hand inequality can be written as

$$\frac{x^2 + nx - 2}{x^2 - 3x + 4} + 1 \ge 0$$

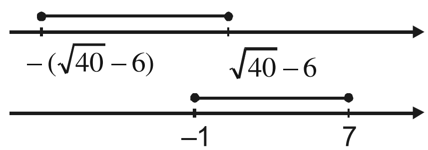
$\Rightarrow \quad \dfrac{2x^2 + (n-3)x + 2}{x^2 - 3x + 4} \geq 0$

$\Rightarrow \quad 2x^2 + (n-3)x + 2 \geq 0$

The above inequality will be true $\forall\ x \in R$ if $D \leq 0$ which gives $-1 \leq n \leq 7$

Drawing the number lines for inequalities (1), (2) and taking their intersection gives

$x \in [-1, \sqrt{40} - 6]$



➤ **Example 2.** For what values of the parameter k in the inequality $\left|\dfrac{x^2 + kx + 1}{x^2 + x + 1}\right| < 3$ satisfied for all real values of x ?

➤ ***Solution*** $\left|\dfrac{x^2 + kx + 1}{x^2 + x + 1}\right| < 3$

$\Rightarrow \quad -3 < \dfrac{x^2 + kx + 1}{x^2 + x + 1} < 3$

$\Rightarrow$ since $x^2 + x + 1 > 0$,

$\therefore \quad -3(x^2 + x + 1) < x^2 + kx + 1 < 3(x^2 + x + 1)$

$\therefore \quad 4x^2 + (k + 3)x + 4 > 0$ ...(1)

and $2x^2 - (k - 3)x + 2 > 0$ ...(2)

Since $4 > 0$ and $2 > 0$

the inequality (1) will be valid

if $(k + 3)^2 - 4.4.4 < 0 \Rightarrow -11 < k < 5$ ...(3)

and the inequality (2) will be valid

if $(k - 3)^2 - 4.2.2. < 0 \Rightarrow -1 < k < 7$ ...(4)

The conditions (3) and (4) will hold simultaneously if $-1 < k < 5$.

Maximum and minimum values of a rational function

Consider $y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}$ ...(i)

To find maximum and minimum values which y can take

cross multiply (i) to get :

$y(px^2 + qx + r) = ax^2 + bx + c$

$\Rightarrow \quad (a - py)x^2 + (b - qy)x + (c - ry) = 0$

Since x is real, $D \geq 0$

$\Rightarrow \quad (b - qy)^2 - 4(a - py)(c - ry) \geq 0$

On solving this inequality we will get the values which y can take.

Case-I :

$y \in [A, B]$

If y can take values between A and B, then,

Maximum value of y = y_{max} = B,

Minimum value of y = y_{min} = A.

Case-II :

$y \in (-\infty, A] \cup [B, \infty)$

If y can take values outside A and B, then,

Maximum value of y = $y_{max} = \infty$ i.e. not defined.

Minimum value of y = $y_{min} = -\infty$ i.e. not defined.

Case-III :

$y \in (-\infty, \infty)$ i.e. $y \in R$

If y can take all values, then

Maximum value of y = $y_{max} = \infty$ i.e. not defined.

Minimum value of y = $y_{min} = -\infty$ i.e. not defined

➤ **Example 3.** If x be a real, show that the expression $\dfrac{x^2 + 2x - 11}{x - 3}$ can take all values which do not lie in the open interval (4, 12).

➤ ***Solution*** Let $y = \dfrac{x^2 + 2x - 11}{x - 3}$

Writing this as a quadratic equation in x, we have

$x^2 + x(2 - y) + (3y - 11) = 0$ ....(1)

The values of x and y are related by this equation and for each value of y, there is a value of x, which is a root of this quadratic equation. In order that this x (or root) is real, discriminant ≥ 0.

$(2 - y)^2 - 4(3y - 11) \geq 0$

$y^2 - 16y + 48 \geq 48$

$(y - 4)(y - 12) \geq 0$

$\therefore \quad y \leq 4$ or $y \geq 12$

Hence y (or the given expression) does not take any value between 4 and 12.

Note If any value between 4 and 12, say 5, is given for y this equation (1) becomes $x^2 - 3x + 4 = 0$ whose roots are imaginary.

➤ **Example 4.** Show that, by giving real values to x, $\dfrac{4x^2+36x+9}{12x^2+8x+1}$ can be made to assume any real value.

Put $\lambda = \dfrac{4x^2+36x+9}{12x^2+8x+1}$

then $x^2(4-12\lambda) + (36-8\lambda)x + 9 - \lambda = 0$.

Now in order that x may be real it is necessary and sufficient that

$(36-8\lambda)^2 - 4(4-12\lambda)(9-\lambda) > 0,$

or that $\lambda^2 - 8\lambda + 72 > 0,$

or $(1-4)^2 + 56 > 0,$

which is clearly true for all real values of λ. Thus we can find real values of x corresponding to any real value of λ.

➤ **Example 5.** Find the range of the expression

$$y = \frac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5},$$

for all permissible values of θ.

➤ ***Solution*** Let $x = \tan\theta$, where $x \in R$ for all permissible values of θ.

$\Rightarrow \quad y = \dfrac{x^2-2x-8}{x^2-4x-5}$

$\Rightarrow \quad x^2y - 4xy - 5y = x^2 - 2x - 8$

$\Rightarrow \quad (y-1)x^2 + 2x(1-2y) + 8 - 5y = 0$

$\therefore \quad x \in R$ hence $D \geq 0$

$\Rightarrow \quad 4(1-2y)^2 - 4(y-1)(8-5y) \geq 0$

$\Rightarrow \quad (4y^2 - 4y + 1) - (13y - 8 - 5y^2) \geq 0$

$\Rightarrow \quad 9y^2 - 17y + 9 \geq 0$ ....(1)

since coefficient of $y^2 > 0$

and $D = 289 - 324 < 0$

Hence (1) is always true.

Therefore range of y is $(-\infty, \infty)$

➤ **Example 6.** Find the values of m for which the expression :

$\dfrac{2x^2-5x+3}{4x-m}$ can take all real values for

$x \in R - \left\{\dfrac{m}{4}\right\}$

➤ ***Solution*** Let $y = \dfrac{2x^2-5x+3}{4x-m}$

$\Rightarrow \quad 2x^2 - (4y+5)x + 3 + my = 0$

$\Rightarrow \quad$ Ax x is real , discriminant ≥ 0

$\Rightarrow \quad (4y+5)^2 - 8(3+my) \geq 0$

$\Rightarrow \quad 16y^2 + (40-8m)y + 1 \geq 0$

A quadratic in y is non-negative for all values of y if coefficient of y^2 is positive and discriminant ≤ 0.

$\Rightarrow \quad (40-8m)^2 - 4(16)(1) \leq 0$

$\Rightarrow \quad (5-m)^2 - 1 \leq 0$

$\Rightarrow \quad (m-5-1)(m-5+1) \leq 0$

$\Rightarrow \quad (m-6)(m-4) \leq 0$

$\Rightarrow \quad m \in [4, 6]$

Consider $m = 4$

$y = \dfrac{(2x-3)(x-1)}{4(x-1)}$

$y = \dfrac{2x-3}{4}, \qquad x \neq 1$

Since $\quad x \neq 1, y \neq -\dfrac{1}{4}$

Hence y cannot take all real values for m = 4. So m = 4 is not acceptable.

Similarly m = 6 is to be rejected.

So for the given expression to take all real values, m should take values : $m \in (4, 6)$.

CAUTION In such problems the end-points of the result are rejected, in general due to cancellation of common factor from numerator and denominator and the resulting expression being incapable of assuming all real values.

➤ **Example 7.** Find the range of the expression

$$y = \frac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{\cot^2\theta + 1},$$

for all permissible values of θ.

➤ ***Solution*** Let $x = \cot^2\theta + 1$, where $x \in [1, \infty)$ for all permissible values of θ.

Then

$$y = \frac{(x+4)(x+9)}{x} = x + 13 + \frac{36}{x}$$

$$= \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)^2 + 25$$

Hence the range of y is $[25, \infty)$

Practice Problems

1. Given $y = \dfrac{x^2 + 2}{x^2 - 1}$, determine the values of y for which x is real .
2. Prove that $\left|\dfrac{12x}{4x^2+9}\right|$ £ 1 for all real values of x, the equality being satisfied only if $|x| = 3/2$.
3. If x is real, show that $\dfrac{2x^2 - 3x + 2}{2x^2 + 3x + 2}$ lies between 1/7 and 7.
4. If x be real and $0 < b < c$ show that $\dfrac{x^2 - bc}{2x - b - c}$ cannot lie between b and c.
5. For real values of x, prove that $\dfrac{11x^2 + 12x + 6}{x^2 + 4x + 2}$ cannot lie between –5 and 3.
6. Show that, if x be real $\dfrac{x^2 - 6x + 5}{x^2 + 2x + 1}$ can never be less than $-\dfrac{1}{3}$.
7. If x is real, show that the expression $\dfrac{4x^2 + 36x + 9}{12x^2 + 8x + 1}$ can have any real value.
8. Show that the expression $\dfrac{mx^2 + 3x - 4}{m + 3x - 4x^2}$ will be capable of all values when x is real, provided that m has any value between 1 and 7.
9. Find all values of ‘a’ for which $\dfrac{ax^2 - 7x + 5}{5x^2 - 7x + a}$ takes all real values.
10. If $\dfrac{mx^2 + 3x + 4}{x^2 + 3x + 4} < 5$ for all $x \in R$, find possible values of m.

2.15 RESOLUTION OF A SECOND DEGREE EXPRESSION IN X AND Y

Let us find the condition that a quadratic function of x and y may be resolved into two linear factors. This is of great importance in analytical geometry.

Denote the function by f(x, y) where

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

If we write this in descending powers of x, and equate it to zero , we get

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0.$$

Solving this quadratic in x we have

$$x = \frac{-(hy + g) \pm \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a},$$

or $\quad ax + hy + g$

$$= \pm\sqrt{y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)}\,.$$

Now in order that f(x, y) may be the product of two linear factors of the form px + qy + r, the quantity under the radical sign must be a perfect square ; hence

$$(hg - af)^2 = (h^2 - ab)(g^2 - ac).$$

Transposing and dividing by a, we obtain

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

CAUTION We note that f(x, y) can be written as a product of two linear factors with real coefficients if apart from the above condition we have either (i) $h^2 - ab > 0$, or (ii) $h^2 - ab = 0$, $hg - af = 0$, $g^2 - ac \geq 0$

➤ **Example 1.** Find the values of k for which the expression

$12x^2 - 10xy + 2y^2 + 11x - 5y + k$ is the product of two linear factors.

Solution Rearrange as a quadratic equation in x.

$$12x^2 + x(11 - 19y) + (2y^2 - 5y + k) = 0$$

Solving

$$x = \frac{(10y-11) \pm \sqrt{(10y-11)^2 - 48(2y^2 - 5y + k)}}{24}$$

The factors will be linear only if

$(10y - 11)^2 - 48(2y^2 - 5y + k)$ is a perfect square and $h^2 - ab > 0$

$\Rightarrow$ if $4y^2 + 20y + (121 - 48k)$ is a perfect square and

$(-5)^2 - 12.2 = 1 > 0$

$\Rightarrow$ discriminant $400 - 16(121 - 48k) = 0$

(i.e.) if $k = 2$.

➤ **Example 2.** Find whether the expression $x^2 + y^2 + 4x + 4$ can be resolved into the product of two linear factors.

➤ ***Solution*** Rearrange as a quadratic equation in x.

$$x^2 + 4x + y^2 + 4 = 0$$

Solving

$$x = \frac{-4 \pm \sqrt{16 - 4(y^2+4)}}{2} = -2 \pm \sqrt{-y^2}$$

$$= -2 \pm iy$$

We can see that the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

is satisfied but the given expression cannot be resolved into product of two linear factors with real coefficients. This happened because $h^2 - ab < 0$.

➤ **Example 3.** Show that the equation $x^2 - xy + y^2 = 4(x + y - 4)$ will not be satisfied for any real value of x and y except x = 4 and y = 4.

➤ ***Solution*** Rewriting the given equation as a quadratic in y

$$y^2 - y(x + 4) + (x^2 - 4x + 16) = 0$$

Since y is real, discriminant ≥ 0

$(x + 4)^2 - 4(x^2 - 4x + 16) \geq 0$

$\Rightarrow$ $-3x^2 + 24x - 48 \geq 0$

$\Rightarrow$ $x^2 - 8x + 16 \leq 0$, on division by (-3)

$(x - 4)^2 \leq 0$

But $(x - 4)^2$ cannot be less than zero

$\therefore$ $(x - 4)^2$ must be equal to zero. $\therefore$ $x = 4$

Similarly we can prove that $y = 4$.

Note that an interchange of x and y does not alter the equation.

➤ **Example 4.** If x, y be real and $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, show that $x \in [3, 6]$, $y \in [1, 10]$

➤ ***Solution*** If we write the given equation as a quadratic in x we get

$$9x^2 + x(2y - 92) + y^2 - 20y + 244 = 0$$

Since x is real we must have

$(2y - 92)^2 - 36(y^2 - 20y + 244) \geq 0$ or $(y - 46)^2 - 9(y^2 - 20y + 244) \geq 0$

$y^2 - 11y + 10 \leq 0 \Rightarrow (y-1)(y-10) \leq 0 \Rightarrow y \in [1, 10]$

Again if we write the given equation as a quadratic in y we get

$$y^2 + y(2x - 20) + 9x^2 - 92x + 244 = 0$$

As before $(2x - 20)^2 - 4(9x^2 - 92 + 244) \geq 0$

$\Rightarrow$ $(x - 10)^2\ (9x^2 - 92x + 244)\ ^3\ 0$ or $x^2 - 9x + 18 \leq 0$

$\Rightarrow$ $x \in [3, 6]$.

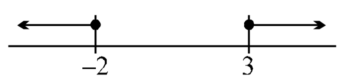
➤ **Example 5.** If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.

➤ ***Solution*** $4y^2 + 4xy + x + 6 = 0$ $\quad y \in R$

$\therefore$ $D \geq 0$

$16x^2 - 16(x + 6) \geq 0$

$\Rightarrow$ $x^2 - x - 6 \geq 0$ $\quad\Rightarrow\quad$ $(x - 3)(x + 2) \geq 0$



$\therefore$ $x \in (-\infty, -2] \cup [3, \infty)$

➤ **Example 6.** Find the greatest and the least real values of x & y satisfying the relation,

$$x^2 + y^2 = 6x - 8y.$$

➤ ***Solution*** writing as a quadratic in y,

$y^2 + 8y + x^2 - 6x = 0$

$y \in R$

$\Rightarrow$ $D \geq 0$

$\Rightarrow$ $x^2 - 6x - 16 \leq 0$

$\Rightarrow$ $-2 \leq x \leq 8$

Similarly range of y can also be obtained :

$-2 \leq x \leq 8$ & $-9 \leq y \leq 1$

➤ **Example 7.** If x, y and z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then show that each of x, y and z lie in the closed interval $\left[\frac{2}{3}, 2\right]$.

➤ ***Solution*** Here, $x^2 + y^2 + (4 - x - y)^2 = 6$

$x^2 + (y - 4)x + (y^2 + 5 - 4y) = 0$

Since x is real $\Rightarrow D \geq 0$

$\Rightarrow \quad (y - 4)^2 - 4(y^2 - 4y + 5) \geq 0$

$\Rightarrow \quad -3y^2 + 8y - 4 \geq 0$

$\Rightarrow \quad -(3y - 2)(y - 2) \geq 0$

$\Rightarrow \quad \frac{2}{3} \leq y \leq 2$

Similarly, we can show that x and z $\in \left[\frac{2}{3}, 2\right]$.

Practice Problems

1. Resolve the expressions into factors :
   (i) $3x^2 + xy - 4y^2 + 8x + 13y - 3$
   (ii) $3x^2 + 2xy - 8y^2 - 7x + 16y - 6$
2. Find the value(s) of λ for which the given expression can be resolved into factors, and for each value of λ resolve the expression into factors:
   (i) $x^2 - xy + 3y - 2 + \lambda(x^2 - y^2)$
   (ii) $x^2 + y^2 - 6y + 4 + \lambda(x^2 - 3y + 2)$.
3. For what values of m will the expression
   $5x^2 + 5y^2 + 4mxy - 2x + 2y + my + m$
   be capable of resolution into two linear factors with real coefficients ?
4. If $(x^2 - 3x - a)y + 4x^2 - 11x + 2a = 0$, find a so that x may be expressed as a rational function of y.
5. If the expression $3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1$ can be resolved into linear factors, prove that p must be one of the roots of $p^2 + 4ap + 2a^2 + 6 = 0$.
6. Show that in the equation
   $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$,
   for every real value of x there is a real value of y, and for every real value of y there is a real value of x.
7. Prove that if the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y, x must lie between 1 and 3 and y must lie between – 1/3 and 1/3.
8. If $5x^2 + 4xy + y^2 - 24x - 10y + 24 = 0$, where x and y are real numbers, show that x must lie between $2 - \sqrt{5}$ and $2 + \sqrt{5}$ inclusive, and y between –4 and 6 inclusive.
9. If $x^2 + xy - y^2 + 2x - y + 1 = 0$, where x and y are real, show that y cannot lie between 0 and $-\frac{8}{5}$, but x can have any real value.

2.16 LOCATION OF ROOTS

In the previous sections, we dealt about the nature of roots of a quadratic equation where we found information about the roots when the coefficients were given.

In the present topic we need to find values of the coefficients when some information about the roots is given. There is one problem to determine the location of roots relative to the point 0 which means finding the signs of roots. Our interest also lies in the location of the roots in a given interval. With respect to this interval we can ask the following questions: Under what conditions : Do both the roots occur in the interval ? Is there exactly one root ? Is there atleast one root ? We can formulate a variety of questions based on these ideas. The student must concentrate on the graphical approach in developing the algebraic conditions associated with location of roots.

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$

and α, β be roots of the equation $f(x) = 0$

i.e. $ax^2 + bx + c = 0$.

In this topic we consider only the real roots of f(x).

For simplicity we can assume that $\alpha \leq \beta$.

where $\alpha = \frac{-b - \sqrt{D}}{2a}$ and $\beta = \frac{-b + \sqrt{D}}{2a}$,

$D = b^2 - 4ac$.

1. Condition for both roots to be positive

   Sum of roots > 0 $\Rightarrow$ $-\frac{b}{a} > 0$

   Product of roots > 0 $\Rightarrow$ $\frac{c}{a} > 0$

   For real roots $\Rightarrow$ $D \geq 0$

2. Condition for both roots to be negative

Sum of roots < 0 $\Rightarrow$ $-\frac{b}{a} < 0$

Product of roots > 0 $\Rightarrow$ $\frac{c}{a} > 0$

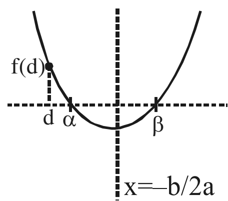
For real roots $\Rightarrow$ $D \geq 0$

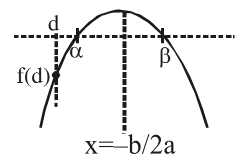
3. Condition for one root negative and other positive i.e. roots opposite in signs.

Product of roots < 0 $\Rightarrow$ $\frac{c}{a} < 0.$

Note that D > 0 is not required since $b^2 - 4ac >$ 0 is satisfied by $\frac{c}{a} < 0.$

4. Condition for both roots to be greater than a given number d.



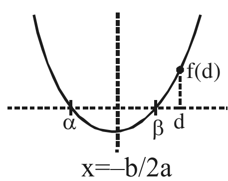


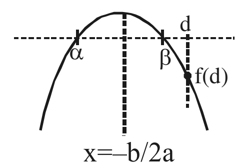
(i) $a > 0$
$f(d) > 0$
$-\frac{b}{2a} > d$
$D \geq 0$

(ii) $a < 0$
$f(d) < 0$
$-\frac{b}{2a} > d$
$D \geq 0$

Combining (i) and (ii) we get $af(d) > 0,\ -\frac{b}{2a} > d,\ D \geq 0$

5. Condition for both roots to be less than a given number d.



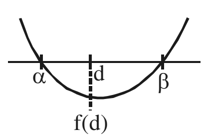


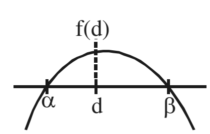
(i) $a > 0$
$f(d) > 0$
$-\frac{b}{2a} < d$
$D \geq 0$

(ii) $a < 0$
$f(d) < 0$
$-\frac{b}{2a} < d$
$D \geq 0$

Combining (i) and (ii) we get $af(d) > 0,\ -\frac{b}{2a} < d,\ D \geq 0.$

6. Condition for one root to be less then d and other greater than d. i.e. a given number d lies between the roots.



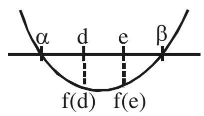


(i) $a > 0$
$f(d) < 0$

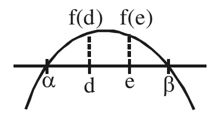
(ii) $a < 0$
$f(d) > 0$

There is no need to take $D > 0$ since the above is sufficient.
Combining (i) and (ii) we get $af(d) < 0$.

7. Condition for one root to be less than d and other greater then e.



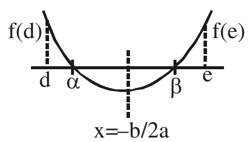
(i) $a > 0$
$f(d) < 0$
$f(e) < 0$



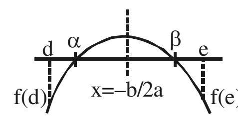
(ii) $a < 0$
$f(d) > 0$
$f(e) > 0$

Combining (i) and (ii) we get $af(d) < 0$, $af(e) < 0$.
Caution : One should not use $f(d) \cdot f(e) > 0$ since it allows other possibilities.

8. Condition for both roots to lie in a given interval (d, e).



(i) $a > 0$
$f(d) > 0$
$f(e) > 0$
$d < \dfrac{-b}{2a} < e$
$D \geq 0$



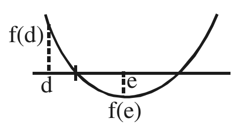
(ii) $a < 0$
$f(d) < 0$
$f(e) < 0$
$d < \dfrac{-b}{2a} < e$
$D \geq 0$

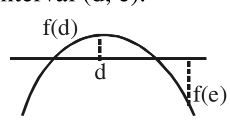
Combining (i) and (ii) we get $af(d) > 0$, $af(e) > 0$, $d < \dfrac{-b}{2a} < e$, $D \geq 0$

9. Condition for exactly one root to lie in a given interval (d, e).

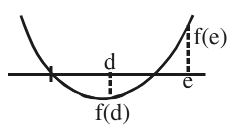
Case-A :

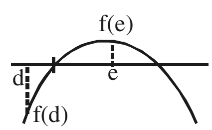
(i) $f(d) > 0$
$f(e) < 0$





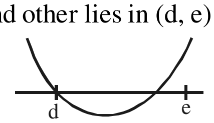
(ii) $f(d) < 0$
$f(e) > 0$





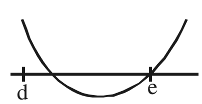
Combining (i) and (ii) we get $f(d) \cdot f(e) < 0$

Case-B :
One root is d and other lies in (d, e)



Solve $f(d) = 0$
Find the other root and check whether it lies in (d, e).

Case–C :
Solve $f(e) = 0$



Find the other root and check whether it lies in (d, e).
Final answer is union of the cases A, B and C.

➤ **Example 1.** If the roots of the quadratic equation,

$$(a^2 - 7a + 13)x^2 - (a^3 - 8a - 1)x + \log_{1/2}(a^2 - 6a + 9) = 0$$

lie on either side of origin, then find the range of value of 'a'.

➤ ***Solution*** $f(0) < 0 \Rightarrow \log_{1/2}(a-3)^2 < 0$

$(a-3)^2 > 1 \Rightarrow (a-4)(a-2) > 0$

$\Rightarrow (-\infty, 2) \cup (4, \infty)$

➤ **Example 2.** Find the range of values of a for which the equation $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0$ has at least one positive root.

➤ ***Solution*** $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0 \quad D \geq 0$

$\Rightarrow (a-5)^2 - 4\left(a - \frac{15}{4}\right) \geq 0$

$\Rightarrow a^2 - 10a + 25 - 4a + 15 \geq 0$

$\Rightarrow a^2 - 14a + 40 \geq 0$

$\Rightarrow (a-4)(a-10) \geq 0$

$\Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I : When both roots are positive

$D \geq 0, a - 5 > 0, a - \frac{15}{4} > 0$

$\Rightarrow D \geq 0, a > 5, a > \frac{15}{4}$

$\Rightarrow a \in [10, \infty)$

Case II : when exactly one root is positive

$\Rightarrow a - \frac{15}{4} \leq 0, \quad a \leq 0\ \frac{15}{4}$

then finally $a \in (-\infty, 15/4] \cup [10, \infty)$

➤ **Example 3.** Find the set of values of 'p' for which the quadratic equation,

$(p-5)x^2 - 2px - 4p = 0$ has atleast one positive root.

➤ ***Solution*** For real roots

$D \geq 0 \Rightarrow p(p-4) \geq 0; \quad p \neq 5$

$\Rightarrow (-\infty, 0] \cup [4, 5) \cup (5, \infty)$

For both non positive roots sum ≤ 0;

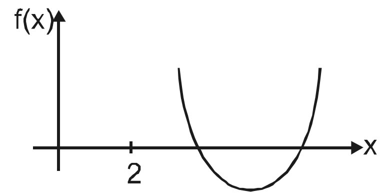
product ≥ 0 & $D \geq 0 \Rightarrow [4, 5)$

Hence for atleast one positive root

$p \in (-\infty, 0] \cup (5, \infty)$

➤ **Example 4.** Consider the quadratic equation $x^2 - (m-3)x + m = 0$ and answer the questions that follow.

(a) Find values of m so that both the roots are greater than 2.



$D \geq 0$

$\Rightarrow (m-3)^2 = 4m \geq 0$

$\Rightarrow m^2 - 10m + 9 \geq 0$

$\Rightarrow (m-1)(m-9) \geq 0$

$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ .... (i)

$f(2) > 0$

$\Rightarrow 4 - (m-3)2 + m > 0$

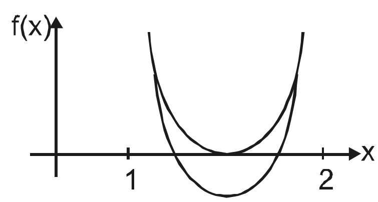
$\Rightarrow m < 10$ .... (ii)

$-\frac{b}{2a} > 2 \Rightarrow \frac{m-3}{2} > 2$

$\Rightarrow m > 7$ .... (iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

(b) Find the values of m so that both roots lie in the interval (1, 2)



$D \geq 0 \Rightarrow m \in (-\infty. 1] \cup [9, \infty)$

$f(1) > 0 \Rightarrow 1 - (m-3) + m > 0$

$\Rightarrow 4 > 0 \Rightarrow m \in R$

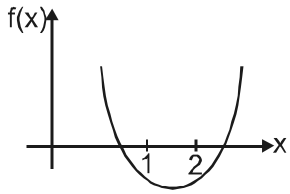
$f(2) > 0 \quad \Rightarrow \quad m < 10$

$1 < -\dfrac{b}{2a} < 0 \quad \Rightarrow \quad 1 < \dfrac{m-3}{2} < 2$

$\Rightarrow \quad 5 < m < 7$

Intersection gives $m \in \phi$

(c) One root is greater than 2 and other smaller than 1

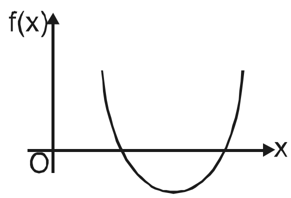


$f(1) < 0 \quad \Rightarrow \quad 4 < 0 \quad \Rightarrow \quad m \in \phi$

$f(2) < 0 \quad \Rightarrow \quad m < 10$

Intersection gives $\quad m \in \phi$

(d) Find the value of m for which both roots are positive.



$D \geq 0 \Rightarrow \quad m \in (-\infty, 1] \cup [9, \infty)$

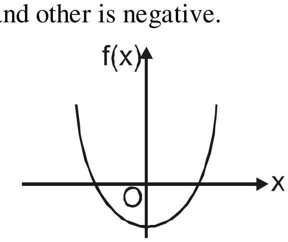
$f(0) > 0 \quad \Rightarrow \quad m > 0$

$-\dfrac{b}{2a} > 0 \quad \Rightarrow \quad \dfrac{m-3}{2} > 0$

$\Rightarrow \quad m > 3$

Intersection gives $m \in [9, \infty)$

(e) Find the values of m for which one root is positive and other is negative.



$f(0) < 0 \quad \Rightarrow \quad m < 0$

(f) Roots are equal in magnitude and opposite in sign.

Sum of roots = 0 $\Rightarrow \quad m = 3$

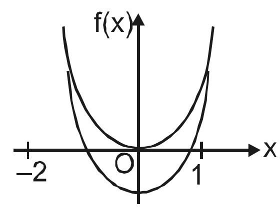
and $f(0) < 0 \quad \Rightarrow \quad m < 0$

$\therefore \quad m \in \phi$

➤ **Example 5.** Find all the values of 'a' for which both the roots of the equation.

$(a-2)x^2 + 2ax + (a+3) = 0$ lies in the interval $(-2, 1)$.

➤ ***Solution***



$(a-2)\, f(-2) > 0$

$\Rightarrow \quad (a-2)[(a-2)\,4 - 4a + a + 3] > 0$

$\Rightarrow \quad (a-2)(a-5) > 0$

$\Rightarrow \quad a < 2, a > 5$

$(a-2)f(1) > 0$

$\Rightarrow \quad (a-2)(4a+1) > 0$

$\Rightarrow \quad a < -\dfrac{1}{4}, a > 2$

$D \geq 0$

$\Rightarrow \quad 4a^2 - 4(a+3)(a-2) \geq 0$

$\Rightarrow \quad a \leq 6$

$-\dfrac{b}{2a} < 1 \quad \Rightarrow \quad \dfrac{2(a-1)}{a-2} > 0$

$\Rightarrow \quad a \in (-\infty, 1) \cup (2, \infty)$

$-2 < -\dfrac{b}{2a} \quad \Rightarrow \quad \dfrac{-2a}{2(a-2)} > -2 \quad \Rightarrow \dfrac{a-4}{a-2} > 0$

$\Rightarrow \quad a \in (-\infty, 2) \cup (4, \infty)$

Complete solution is $a \in \left(-\infty, -\dfrac{1}{4}\right) \cup (5, 6]$

➤ **Example 6.** Find the values of a for which one root of equation $(a-5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2.

➤ ***Solution*** Let $f(x) = (a-5)x^2 - 2ax + a - 4$ ($a \neq 5$) as 1 and 2 lie between the roots of $f(x) = 0$

Consider $(a-5)\,f(1) < 0 : (a-5)(a-5-2a+a-4) < 0$

$\Rightarrow \quad (a-5)(-9) < 0$

$\Rightarrow \quad a-5 > 0$

$\Rightarrow \quad a \in (5, \infty)$ ...(1)

Consider $(a-5)\,f(2) < 0 : (a-5)(4(a-5) - 4a + a - 4) < 0$

$\Rightarrow \quad (a-5)(a-24) < 0$

$\Rightarrow \quad a \in (5, 24)$ ...(2)

Hence, the values of a satisfying (1) and (2) at the same time are $a \in (5, 24)$

➤ **Example 7.** Find the values of a so that the equation

$x^2 + (3-2a)x + a = 0$ has exactly one root in $(-1, 2)$.

➤ ***Solution***

(A) $f(-1)\,.\,f(2) < 0 \quad \Rightarrow \quad (3a-2)(10-3a) < 0$

$$\Rightarrow \quad a \in \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{10}{3}, \infty\right).$$

(B) Let $f(-1) = 0 \Rightarrow 3a - 2 = 0 \Rightarrow a = \frac{2}{3}$

$$\alpha\beta = a \Rightarrow \quad -1\,.\,\beta = \frac{2}{3} \Rightarrow \beta = -\frac{2}{3} \in (-1, 2)$$

Hence $a = \frac{2}{3}$ is acceptable.

(C) Let $f(2) = 0 \Rightarrow 10 - 3a = 0 \quad \Rightarrow a = \frac{10}{3}$

$$\alpha\beta = a \Rightarrow 2\beta = \frac{10}{3} \Rightarrow \beta = \frac{5}{3} \in (-1, 2)$$

Hence $\quad a = \frac{10}{3}$ is also acceptable.

Finally, $\quad a \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{10}{3}, \infty\right)$

➤ **Example 8.** For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$, lies between 1 and 2.

➤ ***Solution*** Since exactly one root of the given equation lies between 1 and 2

we have $\quad f(1)\,f(2) < 0$

Here $f(x) = 2^a x^2 - 4^a x + 2^a - 1$

$\therefore \quad (2^a - 4^a + 2^a - 1)(4.2^a - 2.4^a + 2^a - 1) < 0$

$\Rightarrow \quad (4^a - 2.2^a + 1)(4.4^a - 5.2^a + 1) < 0$

$\Rightarrow \quad (2^a - 1)^2 (2.2^{2a} - 5.2^a + 1) < 0$

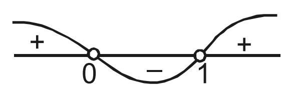
$\Rightarrow \therefore 2.2^{2a} - 5.2^a + 1 < 0 \quad \frac{1}{2} < 2^a < 1$

$\Rightarrow \quad \log_2 (1/2) < a < \log_2 1$

$\Rightarrow \quad -1 < a < 0 \qquad \therefore \qquad a \in (-1, 0)$

➤ **Example 9**. Find all values of a for which the equation $2x^2 - 2(2a+1)x + a(a+1) = 0$ has two roots, one of which is greater than a and the other is smaller than a.

➤ ***Solution*** Here coefficient of x^2 is positive



Let $\quad f(x) = 2x^2 - 2(2a+1)x + a(a+1)$

$\therefore \quad f(a) = 2a^2 - 2(2a+1)a + a(a+1)$

$= -a^2 - a < 0$

$\Rightarrow \quad -a(a+1) < 0 \qquad \Rightarrow \qquad a(a+1) > 0$

$a \in (-\infty, -1) \cup (0, \infty)$

➤ **Example 10.** Find the values of the parameter k for which the equation $x^4 - (k-3)x^2 + k = 0$ has (i) four real roots (ii) exactly two real roots (iii) no real root

➤ ***Solution*** The equation $x^4 - (k-3)x^2 + k = 0$ ...(1)

can be transformed to $t^2 - (k-3)t + k = 0$ ...(2)

where $x^2 = t$

(i) Now the equation (1) will have four real roots if the equation (2) has both roots as positive and distinct. These values of k are included in the interval $(9, \infty)$.

Again (1) will have exactly two real roots if either

(a) (2) has equal roots and this equal root is positive (since $t = x^2$)

(b) (2) has one positive and one negative root.

Now (2) has equal roots if $k = 1$ and $k = 9$

(ii) $k = 1, t = -1 \Rightarrow$ No real roots and $k = 9, t = 3 \Rightarrow$

$x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

Thus at $k = 9$ the original equation has exactly two roots.

Again (2) has one positive and one negative root if $k \in (-\infty, 0)$. Thus the required values of k are 9 and the ones lying in the interval $(-\infty, 0)$.

(iii) Equation (1) will not have and real roots if equation (2) has either no real roots or has both roots negative. Now (2) has no real root if $m \in (1, 9)$ and has both roots negative if $k \in (0, 1)$. Thus required values of k are contained in the interval (0, 9).

➤ **Example 11.** Solve the equation
$9^{-|x-2|} - 4.3^{-|x-2|} - a = 0$ for every real number a.

➤ ***Solution*** Let $y = 3^{-|x-1|}$ and noting that $0 < 3^{-|x-2|} \le 1$ for every x, we get the equation

$$y^2 - 4y - 1 = 0$$

for which we have to find the roots lying in the interval $0 < y \le 1$. The abscissa of the vertex of $f(y) = y^2 - 4y - a$ is equal to 2 so that if the quadratic has roots, then the greater root exceeds 2 and does not interest us. It therefore remains to write down the condition under which there is exactly one root in the interval $0 < y \le 1$.

First of all, $y = 1$ is a root when $a = -3$. Further more, there is exactly one root in the interval $0 < y < 1$ if $f(0).f(1) < 0 \Rightarrow -a(-a-3) < 0 \Rightarrow -3 < a < 0$.

Thus, the equation has exactly one root in the interval

$0 < y \le 1$ for $-3 \le a < 0$.

This is the smaller root $y = 2 - 2\sqrt{4+a}$. Now solving the equation $3^{-|x-2|} = 2 - \sqrt{4+a}$, which for the values of a thus found has a solution , we get

$$|x-2| = -\log_3 (2 - \sqrt{4+a} ,$$

$$x_{1,2} = 2 \pm \log_2 (2 - \sqrt{4+a}) \text{ for } -3 \le a < 0$$

Practice Problems

1. Find all values of p for which the roots of the equation $(p-3)x^2 - 2px + 5p = 0$ are positive.
2. For what values of p does the equation $2x^2 - (p^3 + 8p - 1)x + p^2 - 4p = 0$ possess roots of opposite signs?
3. For what value of m will the equation $\dfrac{x^2 - bx}{ax + c} = \dfrac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign ?
4. If $0 < m < 3$, then show that the roots of the equation $(m-2)x^2 - (8-2m)x - (8-3m) = 0$ are real. Find the range of values of m for which one root is positive and the other is negative.
5. Find all values of m for which the equation $m \in R, m \ne -1, (1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ gives roots according to the following conditions:
   (i) Both roots are
   (a) Imaginary (b) Equal
   (c) Real and distinct (d) Positive
   (e) Negative
   (ii) Roots are opposite in sign
   (iii) Roots are equal in magnitude
   (iv) Roots are equal is magnitude but opposite in sign
   (v) Atleast one root is positive.
   (vi) Atleast one root is negative.
   (vii) Roots are reciprocal to each other
   (viii) Negative root is greater than positive root numerically
   (ix) Positive root is greater than negative root numerically
6. For what values of a, does the equation $ax^2 - (a+1)x + 3 = 0$ have roots lying between 1 and 2?
7. For which values of a is one of the roots of the polynomial $(a^2 + a + 1)x^2 + (a-1)x + a^2$ greater than 3 and the other less than 3?
8. Find all the values of a for which the roots of the equation $x^2 + x + a = 0$ exceed a.
9. For what values of 'a' does the equation $2\log_3^2 x - |\log_3 x| + a = 0$ possess four solutions?
10. Find all the values of the parameter a for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ lie between 0 and 3 .

11. Find all values of m for which the equation $m \in R, m \neq -1, (1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ gives roots according to the following conditions:

(i) Exactly one root in the interval (2, 3)

(ii) One root smaller than 1 and other root greater than 1

(iii) Both roots smaller than 2

(iv) Atleast one root in the interval (2, 3)

(v) Atleast one root greater than 2

(vi) Roots such that both 1 and 2 lie between them.

(vii) One root in (1, 2) and other root in (2, 3)

12. If the quadratic equation $ax^2 + bx + c = 0$ has roots of opposite sign lying in the interval (–2, 2), then prove that $1 + \frac{c}{4a} - \left|\frac{b}{2a}\right| > 0$.

13. For what values of a do the real roots, x_1 & x_2, of the equation $x^2 - 4ax + 1 = 0$ satisfy $x_1 \geq a$ and $x_2 \geq 0$?

2.17 SOLVING INEQUALITIES BASED ON LOCATION OF ROOT

➤ **Example 1** Find all values of 'k' for which the inequality $k \cdot 4^x + (k-1)2^{x+2} + k > 1$ is satisfied for all $x \in R$.

➤ ***Solution*** Let $2^x = t$ where $t \in (0, \infty)$.

The inequality is $kt^2 + 4(k-1)t + k - 1 > 0$ for $t > 0$

Let $f(t) = kt^2 + 4(k-1)t + k - 1$

Case A :

f(t) is positive for all $t \in R$.

$k > 0$

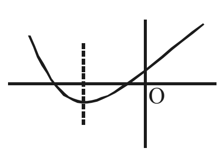
$D < 0 \Rightarrow \quad 4(k-1)^2 - (k-1)k < 0$

$\Rightarrow \quad (k-1)(3k-4) < 0$

$\Rightarrow \quad 1 < k < \frac{4}{3}$.

Case B :

$f(t) > 0$ for $t > 0$



and f(t) may be non-positive for $t \leq 0$.

$a > 0 \quad \Rightarrow \quad k > 0$

$f(0) \geq 0 \quad \Rightarrow \quad k - 1 \geq 0$

$\frac{-b}{2a} \leq 0 \quad \Rightarrow \quad \frac{-4(k-1)}{2k} \leq 0$

$\Rightarrow \quad \frac{k-1}{k} \geq 0 \quad \Rightarrow \quad k < 0$ or $k \geq 1$

There is no need of discriminant.

Taking intersection, we get $k \geq 1$.

Case C :

Let the leading coefficient be zero i.e. $k = 0$

Then $f(t) = -4t - 1$ which is always negative since $t > 0$. The inequality is not satisfied.

Finally, we have $k \in [1, \infty)$.

➤ **Example 2** Find all possible parameters 'a' for which, $f(x) = (a^2 + a - 2)x^2 - (a+5)x - 2$ is non positive for every $x \in [0, 1]$.

➤ ***Solution*** $f(x) = (a+2)(a-1)x^2 - (a+5)x - 2 \leq 0$

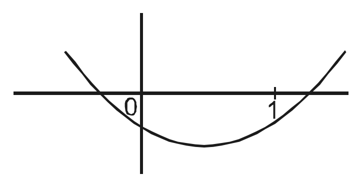
for $a = -2$; $f(x) = -3x - 2$

which is negative in [0, 1]

for $a = 1$; $f(x) = -6x - 2$

which is negative in [0, 1]

Now **Case I** :



for $a^2 + a - 2 > 0$ i.e. $a > 1$ or $a < -2$ — (1)

(i) $f(0) \leq 0$ always true

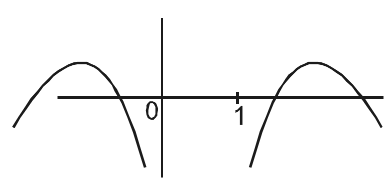
(ii) $f(1) \leq 0$

$\Rightarrow \quad a \in [-3, 3]$ — (2)

from (1) and (2) $a \in [-3, -2) \cup (1, 3]$

Case II $a^2 + a - 2 < 0$

i.e. $-2 < a < 1$



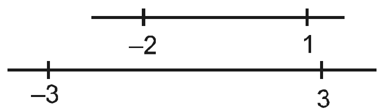
In this case :

(i) $D \geq 0 \quad \Rightarrow \quad (a+2)^2 \geq 0 \quad \Rightarrow \quad a \in R$

(ii) $f(1) < 0$ & $f(0) < 0$ (always true)

$\Rightarrow \quad a \in (-3, 3)$

common solution $(-2, 1)$



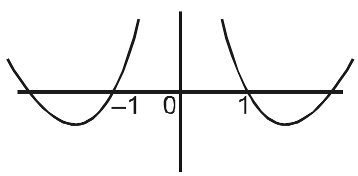
Case III $a < 0$ & $D < 0$

since $D \geq 0 \quad \Rightarrow$ no solution in this case

combining all $a \in [-3, 3]$

➤ **Example 3** Find all values of k for which the inequality, $2x^2 - 4k^2x - k^2 + 1 > 0$ is valid for all real x which do not exceed unity in the absolute value.

➤ ***Solution*** **Case I** when $D < 0$ which gives the result



Case II $f(1) \geq 0$; $D \geq 0$; $-\dfrac{b}{2a} > 1$

or $f(-1) \geq 0$; $D \geq 0$; $-\dfrac{b}{2a} < -1$;

No solution in case II

$$-\frac{1}{\sqrt{2}} < k < \frac{1}{\sqrt{2}}$$

➤ **Example 4** Find the values of a for which the inequality $\tan^2 x + (a + 1)\tan x - (a - 3) < 0$ holds for atleast one $x \in \left(0, \dfrac{\pi}{2}\right)$

➤ ***Solution*** We are required to find the values of a for which

$$\tan^2 x\,(a + 1)\tan x - (a - 3) < 0$$

for some $x \in \left(0, \dfrac{\pi}{2}\right)$

i.e. $y^2 + (a + 1)y - (a - 3) < 0$ for some $y > 0$ [assume $\tan x = y$]

i.e. $f(y) = y^2 + (a + 1)y + (3 - a)$.

Since the coeff. of y^2 is positive the plot of f represents a parabola whose mouth opens upward.

According to the required condition, $f(y) < 0$ for some positive y

$\Rightarrow$ a portion of the parabola must lie in the fourth quadrant

$\Rightarrow$ atleast one root of $f(y) = 0$ is positive.

Case I (roots opposite in sign)

product of roots < 0 i.e. $a > 3$.

Case II (both roots positive)

$$D \geq 0 \Rightarrow (a + 1)^2 + 4(a - 3) \geq 0 \Rightarrow a^2 + 6a - 11 \geq 0$$

$\Rightarrow \quad a \leq -3 - 2\sqrt{5}$ or $a \geq -3 + 2\sqrt{5}$ ...(1)

and product of roots > 0

$\Rightarrow \quad a < 3$ ...(2)

and sum of roots > 0

$\Rightarrow \quad a < -1$ ...(3)

Intersection of inequalities (1), (2) and (3), gives

$$a \leq -3 - 2\sqrt{5}.$$

Now, union of the inequalities obtained in both the cases gives

$$a \in (-\infty, -3, -2\sqrt{5}\,] \cup (3, \infty).$$

Practice Problems

1. Find all real values of m for which the inequality $mx^2 - 4x + 3m + 1 > 0$ is satisfied for all positive x.

3. Find all values of m for which all $x \in [1, 2]$ is a solution of the inequality $x^2 - mx + 1 < 0$.

4. Find all values of a for which $x^2 - x + a - 3 < 0$ for atleast one negative x.

5. Prove that for any value of a the inequality $(a^2 + 3)x^2 + (a + 2)x - 4 < 2$ is true for atleast one negative x.

6. For what values of $m \in R_1$ $m \neq -1$, the function $f(x) = (1 + m)x^2 - 2(1 + 3m)x + (1 + 8m)$ is

(a) positive for all $x \geq 2$
(b) negative for all $x \geq 2$
(c) less than 1 for all $x \in (-\infty, 0)$
(d) greater than 5 for all $x \in (-1, 1)$

7. Find all values of 'a' for which $9^x - a.\ 3^x - a + 5/4 \leq 0$ is satisfied for atleast one $x \in R$.

THEORY OF EQUATIONS

2.18 POLYNOMIALS

An expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2\, x^{n-2} + ... + a_{n-1}x + a_n$$

in which $a_0, a_1, a_2, ..., a_n$ are called coefficients (constants free from x) and n is a non-negative integer, is called a polynomial in x.

If $a_0 \neq 0$, the polynomial is of degree n, a_0x^n is the leading term, and a_0 is the leading coefficient. Also a_n is the constant term.

A polynomial in one variable is written in descending powers of the variable when the exponents on the terms decrease from left to right. Writting with descending powers of x, the polynomial is said to be in standard form.

The n + 1 coefficients can be determined, if the values of the polynomial are known for n + 1 values of x.

Polynomial of two or more variables

If a polynomial contains more than one variable, the degree of a term is the sum of the exponents on all the variables. The degree of the polynomial is the greatest of the degrees of any of its terms.

A polynomial of two variables x and y is the algebraic sum of several terms of the form ax^ry^s where r and s are positive integers. r + s is the degree of the term. For example, the degree of $3x^2\,y^7$ is 2 + 7 or 9. Also $x^5 + 3x^4y + 2xy^3 + 4y^2 - 2y + 1$ is a polynomial of x and y and its degree is 5.

Similarly an expression whose terms are of the form $ax^ry^sz^t$ where r, s and t are positive integer is a polynomial of three variables x, y and z.

Homogeneous polynomial

A polynomial of more than one variable is said to be homogeneous, if the degree of each term is the same. Thus, $2x^7 + 5x^5y^2 - 3x^4y^3 + 4x^2\,y^5$ is a homogeneous polynomial of degree 7 in x and y.

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$ is a homogeneous polynomial of degree 2 in x, y and z.

➤ **Example 1.** If the value of the polynomial $a_0x^2 + a_1x + a_2$ are 10, 19 and 7 for x = 1, 2 and –2 respectively, determine the co-efficients.

➤ ***Solution*** Here $a_0\,.\,1^2 + a_1\,.\,1 + a_2 = 10$,

or $a_0 + a_1 + a_2 = 10$ ...(1)

$a_0\,.\,2^2 + a_1\,.\,2 + a_2 = 19$,

or $4a_0 + 2a_1 + a_2 = 19$ ...(2)

$a_0(-2)^2 + a_1(-2) + a_2 = 7$,

or $4a_0 - 2a_1 + a_2 = 7$ ...(3)

From (1), (2) and (3) we get $a_0 = 2, a_1 = 3, a_2 = 5$.

Identical polynomials

Two polynomials in x are identical if the coefficients of their like powers of x are equal.

Thus, $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}\,x + a_n$

$= b_0x^n + b_1x^{n-1} + b_2x^{n-2} + ...+ b_{n-1}\,x + b_n$

implies that $a_0 = b_0, a_1 = b_1, ..., a_n = b_n$.

➤ **Example 2.** Obtain the condition that $x^3 + 3px + q$ may have a factor of the form $(x - a)^2$.

➤ ***Solution*** If $(x - a)^2$ is a factor we may assume that

$x^3 + 3px + q = (x - a)^2\,(x + b) = (x^2 - 2ax + a^2)\,(x + b)$

$= x^3 + (b - 2a)x^2 + (a^2 - 2ab)x + a^2b$.

Since the two polynomials are identical we have

$b - 2a = 0$ ...(1)

$a^2 - 2ab = 3p$ ...(2)

$a^2b = q$ ...(3)

From (1) b = 2a. Putting b = 2a in (2) $3p = -3a^2$, or, $p = -a^2$.

From (3) $2a^3 = q \Rightarrow 4a^6 = q^2 \Rightarrow 4(-p)^3 = q^2 \Rightarrow 4p^3 + q^2 = 0$.

➤ **Example 3.** Using the method of undetermined coefficients, show that the expression $(x + 1)\,(x + 2)\,(x + 3)\,(x + 4) + 1$ is the square of a trinomial.

➤ ***Solution*** If the given expression is the square of a trinomial, then the following equality is true:

$(x + 1)(x + 2)(x + 3)(x + 4) + 1 = (x^2 + ax + b)^2$, where a and b are the desired coefficients.

On performing the operations indicated in this equality, we compare the coefficients of x^3 and x^2 in both sides. We obtain the system

$$\begin{cases} 2a = 10, \\ a^2 + 2b = 35, \end{cases}$$

Whence we find a = 5 and b = 5. Make sure that for these values of a and b the coefficients of x and x^0 (i.e. constant terms) also coincide. Thus, the given expression is equal to $(x^2 + 5x + 5)^2$.

➤ **Example 4.** Find the factorization of the polynomial $P(x) = (x^2 + x + 1)(x^2 + x + 2) - 12$.

➤ ***Solution*** $P(x) = (x^2 + x + 1)((x^2 + x + 1) + 1) - 12$
$= (x^2 + x + 1)^2 + (x^2 + x + 1) - 12.$

Let $x^2 + x + 1 = y$. We then have $y^2 + y - 12 = (y + 4)(y - 3)$, since the roots of the trinomial $y^2 + y - 12$ are equal to (–4) and 3. Passing from y to x, we get

$P(x) = (x^2 + x + 5)(x^2 + x - 2)$.

Since the trinomial $x^2 + x - 2 = (x - 1)(x + 2)$, $P(x) = (x - 1)(x + 2)(x^2 + x + 5)$.

Polynomial Equation

If a polynomial in x becomes equal to another polynomial for certain values of x, then the algebraical expression of such a relation is called an equation. Any value of x which satisfies the equation is called a root of the equation. Determination of all possible roots is known as the complete solution of the equation.

By bringing all the terms to one side, the equation can be arranged in descending powers of x in the following form:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

Therefore, a polynomial equation is obtained by equating a polynomial to zero. The highest power of x in the equation is called the degree of the equation. An equation remains unchanged if all the terms are divided by the same non-zero number. It is said to be complete if it contains all the powers of x from n to 0; otherwise it is incomplete.

If α be a complex number such that $f(\alpha) = 0$, then α is called a **root or zero** of the polynomial f.

$3x^3 + 4x^2 - 7$ is a polynomial of degree 3 with real coefficients; 1 is a zero of this polynomial.

$x^3 + 2x - i$ is a polynomial of degree 3 with complex coefficients; i is a zero of this polynomial.

➤ **Example 5.** If x = 1 and x = 2 are roots of the equation $x^3 + ax^2 + bx + c = 0$ and a + b = 1, then find the value of b.

➤ ***Solution*** $a + b + c = -1$

$\Rightarrow \quad c = -2$

and $\quad 8 + 4a + 2b + c = 0$

$\Rightarrow \quad 4a + 2b = -6$

$\Rightarrow \quad 2a + b = -3$

$\Rightarrow \quad a = -4, \; b = 5$

hence $\quad a = -4; \; b = 5; c = -2$

2.19 REMAINDER THEOREM

Remainder Theorem

Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x) is divided (x – a), then the remainder is equal to P(a).

Proof :

Since x – a is of first degree, the remainder will be a constant. If we represent the quotient by Q and the remainder by R, then $f(x) = (x - a) \times Q + R$. By substituting a for x, we get the result.

Factor Theorem

Let P(x) be polynomial of degree greater than of equal to 1 and 'a' be a real number such that P(a) = 0, then (x – a) is a factor of P(x). Conversely, if (x – a) is a factor of P(x), then P(a) = 0.

Note: If f(x) has integer coefficients and a is an integer root of f(x) and m is any integer different from a, then a – m divides f(m).

Proof :

On dividing f(x) by x – m we get

$f(x) = (x - m)\, q(x) + f(m)$,

where q(x) has integer coefficients. So for x = a, we get

$0 = f(a) = (a - m)\, q(a) + f(m)$ or $f(m) = (a - m)\, q(a)$.

Hence (a – m) divides f(m).

➤ **Example 1.** Let f(x) be a polynomial, having integer coefficients and let f(0) = 1989 and f(1) = 9891. Prove that f(x) has no integer roots.

➤ ***Solution*** If a is an integer root, then $a \neq 0$ as $f(0) \neq 0$. Also a must be odd since it must divide $f(0) = a_n = 1989$. But $a \neq 1$ as $f(1) \neq 0$. So taking m = 1 in the above Note, we see that the even number (a – 1) divides the odd number f(1) = 9891, a contradiction.

STUDY TIP When one root of an equation can be found by inspection using remainder theorem, the degree of the equation can be lowered by means of division.

In the equation $(a - x)^4 + (x - b)^4 = (a - b)^4$,
since x = a and x = b both satisfy the equation,
$(x - a)(x - b)$ will divide $(a - x)^4 + (x - b)^4 - (a - b)^4$, and as the quotient will be of the second degree, the equation formed by equating it to zero can be solved.

➤ **Example 2.** Find p and q so that (x + 2) and (x – 1) may be factors of the polynomial
$f(x) = x^3 + 10x^2 + px + q$.

➤ ***Solution*** Since (x + 2) is a factor f(–2) must be zero $\therefore\ -8 + 40 - 2p + q = 0$ ...(1)

Since (x – 1) is a factor, f(1) must be zero

$\therefore\quad 1 + 10 + p + q = 0$ ...(2)

From (1) and (2), by solving we get p = 7 and q = –18

➤ **Example 3.** Show that (2x + 1) is a factor of the expression $f(x) = 32x^5 - 16x^4 + 8x^3 + 4x + 5$.

➤ ***Solution*** Since (2x + 1) is to be a factor of

$f(x),\ f\left(-\frac{1}{2}\right)$ should be zero.

$$f\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^5 - 16\left(-\frac{1}{2}\right)^4 + 8\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right) + 5$$

Hence (2x + 1) is a factor of f(x).

➤ **Example 4.** Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

➤ ***Solution*** Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials. Then

$g(x) = x^2 - 3x + 2 \quad = (x - 1)(x - 2)$

In order to prove that f(x) is exactly divisible by & (x), it is sufficient to prove that x – 1 and x – 2 are factors of f(x). For this it is sufficient to prove that f(1) = 0 and f(2) = 0.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow\quad f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2$

and, $f(2) = 2 \times 2^2 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2$

$\Rightarrow\quad f(1) = 2 - 6 + 3 + 3 - 2$ and $f(2) = 32 - 48 + 12 + 6 - 2$

$\Rightarrow\quad f(1) = 8 - 8$ and $f(2) = 50 - 50$

$\Rightarrow\quad f(1) = 0$ and $f(2) = 0$

$\Rightarrow\quad (x - 1)$ and $(x - 2)$ are factors of f(x).

$\Rightarrow\quad g(x) = (x - 1)(x - 2)$ is a factors of f(x).

Hence, f(x) is exactly divisible by g(x).

➤ **Example 5**. Show that $(x - 1)^2$ is a factor of
$x^n - nx + n - 1$.

➤ ***Solution*** Let $f(x) = x^n - nx + n - 1$. Since f(x) = 0, therefore by the factor theorem f(x) is divisible by x – 1.

We can write $f(x) = (x^n - 1) - n(x - 1)$,

$= (x - 1)\{x^{n-1} + x^{n-2} + ... + 1 - n\}$

$= (x - 1)\, g(x)$,

where $g(x) = x^{n-1} + x^{n-2} + ... + 1 - n$.

since g(1) = 0, therefore by the factor theorem x – 1 is a factor of g(x), and consequently f(x) is divisible by $(x - 1)^2$.

➤ **Example 6.** A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by (x – 1); (x + 2) & (x + 1) respectively. Find the remainder, if the polynomial is divided by, $(x^2 - 1)(x + 2)$.

➤ ***Solution*** $f(x) = Q_1(x - 1) + 2 = Q_2(x + 2) + 1$
$= Q_3(x + 1) - 1$

$\Rightarrow\quad f(1) = 2\ ;\quad f(-2) = 1\ ;\quad f(-1) = -1$

again $f(x) = Q_r(x^2 - 1)(x + 2) + ax^2 + bx + c$

Hence $a + b + c = 2\ ;\ 4a - 2b + c = 1$

and $a - b + c = -1$

➤ **Example 7.** Find all cubic polynomials p(x) such that $(x-1)^2$ is a factor of p(x) + 2 and $(x+1)^2$ is a factor of p(x) – 2.

➤ ***Solution*** If $(x-\alpha)$ divides a polynomial q(x) then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since (x – 1) divides p(x) + 2, we get a + b + c + d + 2 = 0.

Hence d = –a – b – c– 2 and

$$p(x) + 2 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$$
$$= (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}.$$

Since $(x-1)^2$ divides p(x) + 2, we conclude that (x – 1) divides $a(x^2 + x + 1) + b(x + 1)$ c. This implies that 3a + 2b + c = 0. Similarly, using the information that $(x + 1)^2$ divides p(x) – 2, we get two more relations : –a + b – c + d – 2 = 0; 3a – 2b + c = 0. Solving these for a, b, c, d, we obtain b = d = 0, and a = 1, c = –3. Thus there is only one polynomial satisfying the given condition : $p(x) = x^3 - 3x$.

➤ **Example 8.** Show that the polynomial

$x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$.

➤ ***Solution*** Let $f(x) = x(x^{n-1} - na^{n-1}) + a^n(n-1)$ then $f(a) = a(a^{n-1} - na^{n-1}) + a^n(n-1) = 0$.

Also $f'(x) = nx^{n-1} - na^{n-1} \Rightarrow f'(a) = 0$

Since both f(a) and f′(a) are zero. The polynomial f(x) must be divisible by $(x-a)^2$.

➤ **Example 9.** Find the remainder when

(i) $x^{100} + x + 2$ is divided by x – 1

(ii) $x^{100} + x + 2$ is divided by $x^2 - 1$

➤ ***Solution***

(i) Let $f(x) = x^{100} + x + 2$ then answer is clearly f(1) = 4.

(ii) Note that the remainder when f(x) is divided by x + 1 is f(–1) = 2

Using f(1) = 4, f(–1) = 2 (remainders) we write

$$\frac{f(x)}{x-1} = g(x) + \frac{4}{x-1}; \frac{f(x)}{x+1} = h(x) + \frac{2}{x+1}$$

where g(x) and h(x) are two polynomials.

On subtracting we get

$$\frac{2f(x)}{x^2-1} = g(x) - h(x) + \frac{2x+6}{x^2-1}$$

or $$\frac{f(x)}{x^2-1} = \frac{1}{2}g(x) - \frac{1}{2}h(x) + \frac{x+3}{x^2-1}$$

⇒ The remainder when f(x) is divided by $x^2 - 1$ is x + 3.

Practice Problems

1. Show by substitution that x – 1, x – 5, x + 2 and x + 4 are factors of $x^4 - 23x^2 - 18x + 40$.
2. Show that $x^4 + 3ax^3 - 7x -$ is divisible by 2x + 3 when $a = \frac{121}{162}$.
3. Prove that $4ax^3 - 2x^2 + 3bx + 5$ is divisible by 2x – 1 and x + 2 if $b = -3\frac{1}{6}$.
4. If f(x) is a rational integral function of x and a, b are unequal, show that the remainder in the division of f(x) by (x – a) (x – b) is [(x – a) f(b) – (x – b) f(a)]/(b – c).
5. For what values of parameters m and n is the polynomial $5x^6 - px^5 + 2qx^2 - 2x + 3$ divisible by $x^2 - 1$.
6. Find a quadratic function of x which shall vanish when $x = -\frac{3}{7}$ and have the values 24 and 62 when x = 3 and x = 4 respectively.
7. If a, b, c, d are in G.P. then prove that $ax^2 + bx^2 + cx + d$ is divisible by $ax^2 + c$.
8. Show that $x^3 + \alpha x^2 + \beta x + \gamma$ will be a perfect cube when $\alpha^2 = 27\gamma$ and $3\alpha\gamma = \beta^2$.
9. Prove that $ax^3 + bx + c$ is divisible by $x^2 + px + 1$ if $a^2 - c^2 = ab$.

2.20 ROOTS OF A POLYNOMIAL EQUATION

Fundamental Theorem of Algebra

Every polynomial equation of degree ≥ 1 with given coefficients (real or imaginary) has atleast one root (real or imaginary). This theorem was discovered by the famous German mathematician Karl Friedrich Gauss.

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ be a polynomial of degree n with the leading coefficient a_0. As a result of the fundamental theorem, the equation f(x) = 0 has a real or imaginary root α_1 so that $f(\alpha_1) = 0$. By the remainder theorem f(x) is divisible by $(x - \alpha_1)$. Therefore, $f(x) = (x - \alpha_1) \times f_1(x)$, where $f_1(x)$ is polynomial of degree n – 1 with the leading coefficient a_0. By the fundamental theorem again, the equation $f_1(x) = 0$ has a real or imaginary root α_2. Therefore, $f_1(\alpha_2) = 0$.

Consequently $f_1(x) = (x - \alpha_2) = (x - \alpha_2) \times f_2(x)$, where $f_2(x)$ is a polynomial of degree n – 2 with the leading coefficient a_0. The same procedure is repeated until we come to a polynomial of the first degree, $f_{n-1}(x)$ with the root α_n and the leading coefficient a_0. So,

$$f_{n-1}(x) = a_0(x - \alpha_n).$$

Combining all these relations, we get

$$f(x) = a_0 (x - \alpha_1)(x - \alpha_2) ..... (x - \alpha_n).$$

From the above relation we conclude that every polynomial in x of degree n can be factorized into n linear factors of the form $(x - \alpha)$. All of them may not be distinct, some may be equal. If x is replaced by any one of $\alpha_1, \alpha_2,....,\alpha_n$ f(x) will reduce to zero and no other value of x, different from $\alpha_1, \alpha_2, ..., \alpha_n$ will make f(x) zero.

Theorem : A polynomial f(x) of the n-th degree cannot vanish for more than n values of x unless all its coefficients are zero.

For otherwise the product of more than n expressions of the form $x - \alpha$ would be a factor of the polynomial. That is to say, a polynomial of the nth degree would have a factor of degree higher than n, which is impossible.

Corollary 1 : If a polynomial in x of degree n vanishes for more than n distinct values of x, it vanishes identically.

For, if $f(x) = a_0(x - \alpha_1)(x - \alpha_2) .... (x - \alpha_n)$ ...(1)

vanishes for $x = \alpha$, different from $\alpha_1, \alpha_2, ..., \alpha_n$, we have $0 = a_0(\alpha - \alpha_1)(\alpha - \alpha_2) ..... (\alpha - \alpha_n)$. Therefore, $a_0 = 0$. Hence from (1) all the coefficients in f(x) are zeros and f(x) vanishes identically.

Corollary 2 : Two polynomials in x of degree n will be identical, if their equality holds for more than n values of x.

Equating the difference of the two polynomials to zero, we get an equation of degree n, which is satisfied by more than n values of x, and so all the coefficients will be zeros. Hence the original polynomials must be identically equal.

➤ **Example 1.** Form a fourth degree polynomial equation whose roots are $0, \pm a, \frac{c}{b}$.

➤ ***Solution*** The equation has to be satisfied by

$$x = 0, x = a, x = -a, x = \frac{c}{b};$$

therefore it is $x(x + a)(x - a)\left(x - \frac{c}{b}\right) = 0$;

that is, $x(x^2 - a^2)(bx - c) = 0$,

or $bx^4 - cx^3 - a^2bx^2 + a^2cx = 0$.

Identity

f(x) = 0 is said to be an identity in x if it is satisfied by all values of x in the domain of f(x)

Thus an identity in x is satisfied by all values of x whereas an equation in x is satisfied by some particular values of x.

(i) $(x + 2)^2 = x^2 + 4x + 4$ is an identity in x.

Here highest power of x in the given relation is 2 and this relation is satisfied by three different values x = 0, x = 1 and x = –1 and hence it is an identity because a polynomial equation of n th degree cannot have more than n distinct roots.

(ii) $x^9 - 2x^7 + 5 = 0$ is an equation in x because it is not satisfied by x = 0.

Equivalent equations : Two equations are said to be equivalent if they have the same roots.

STUDY TIP The symbol ≡ is used to distinguish an identity from an equation.

In an identity in x coefficients of similar powers of x on the two sides are equal.

Thus if $ax^3 + bx^2 + cx + d = 2x^3 - 5x^2 + 3x - 6$ be an identity in x, then a = 2, b = –5, c = 3, d = –6.

➤ **Example 2**. Solve the equation

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+ c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^3$$

➤ ***Solution*** the equation is satisfied by $x = a$, by $x = b$, and by $x = c$; and the equation is not an identity, since the coefficient of x^3 is not zero.
Hence the roots of the cubic are a, b, c.

➤ **Example 3.** Show that, if

$(a-\alpha)^2 x + (a-\beta)^2 y + (a-\gamma)^2 z = (a-\delta)^2,$
$(b-\alpha)^2 x + (b-\beta)^2 y + (b-\gamma)^2 z = (b-\delta)^2,$
$(c-\alpha)^2 x + (c-\beta)^2 y + (c-\gamma)^2 z = (c-\delta)^2,$

then will

$(d-\alpha)^2 x + (d-\beta)^2 y + (c-\gamma)^2 z = (d-\delta)^2$, where d has any value whatever.

➤ ***Solution*** The equation

$$(X-\alpha)^2 x + (X-\beta)^2 y + (X-\gamma)^2 z = (X-\delta)^2$$

is a quadratic equation is X, and it has the three roots a, b, c. It is therefore satisfied when any other quantity d is put for X.

Equal Roots

A polynomial of degree n has n linear factors and a number of them may be repeated. The factor $(x-\alpha)$ may occur twice, thrice, or more times but not more than n times. In this case, the equation $f(x) = 0$ said to have n roots, two or more being equal to one another. If $f(x) = a_0(x-\alpha)^r (x-\beta)......(x-\delta)$, then α is called a multiple root of $f(x) = 0$ and r is called the multiplicity of α.

➤ **Example 4**. Find the zeros of the polynomial $(x-1)(x+2)^2(x-3)^2$ and give the multiplicity of each.

➤ ***Solution*** The zeros are 1, –2, –2, 3, 3, i.e., 1 is a single root, and –2 and 3 are roots with multiplicity two each.

Imaginary Roots

If a complex number of the form $\alpha + i\beta$, $\beta \neq 0$, be a root of $f(x) = 0$ whose coefficients are all real, then $\alpha - i\beta$ will be another root of $f(x) = 0$, $(i^2 = -1)$

Therefore f(x) is divisible by $(x-\alpha-i\beta)(x-\alpha+i\beta)$, that is, by $(x-\alpha)^2+\beta^2$.
Thus a polynomial in x with real coefficients can be resolved into factors which are linear or quadratic functions of x with real coefficients.

Note For a cubic equation $ax^3 + bx^2 + cx + d = 0$, whose roots are α, β, γ either all the three roots are real or one root is real (the other two roots being imaginary).

CAUTION In an equation with complex coefficients, complex roots may not occur in conjugate pairs. For example, the roots of the quadratic equation $x^2 - 7xi - 12 = 0$ are 3i and 4i. Here 3i is a root but its complex conjugate, –3i, is not a root.

➤ **Example 5.** Solve the equation

$2x^3 - 15x^2 + 46x - 42 = 0$, having given that one root is $3 + \sqrt{-5}$.

➤ ***Solution*** Since $3 \pm \sqrt{-5}$ are roots of the equation

$$(x-3-\sqrt{-5})(x-3+\sqrt{-5})$$

must be a factor of the left-hand side of the equation, which may be written

$$\{(x-3)^2+5\}(2x-3)=0.$$

The roots required are $3 \pm \sqrt{-5}$, $\frac{3}{2}$.

➤ **Example 6** Form the equation of lowest degree with real coefficients which has $2 + 3i$, $3 - 2i$ as two of its roots.

➤ ***Solution*** Since in an equation with real coefficients, complex roots occur in conjugate pairs, therefore the required equation must have at least four roots, namely, $2 \pm 3i$, $3 \pm 2i$, i.e. it must have

$$\{x-(2+3i)\}\{x-(2-3i\}\{x-(3+2i)\}\{x-(3-2i)\}$$

as a factor,

i.e., it must have $\{(x-2)^2+9\}\{(x-3)^2+4\}$ as a factor.

Since the coefficients are real, therefore $\{(x-2)^2+9)\}\{(x-3)^2+4\} = 0$

is the equation of the lowest degree with real coefficients having 2 + 3i, 3 – 2i as two of the roots.

Roots as Quadratic Surds

In an algebraic equation with rational coefficients, the roots in the form of a quadratic surd occur in conjugate pairs. Let the quadratic surd $p + \sqrt{q}$ be a root of the algebraic equation f(x) = 0 whose coefficients are all rational. Then $p - \sqrt{q}$ is also a root of the equation.

Note

If $\sqrt{p}$ and $\sqrt{q}$ are two dissimilar quadratic surds and $\sqrt{p}$ and $\sqrt{q}$ is a root of f(x) = 0 with rational coefficients then $\pm\sqrt{p} \pm \sqrt{q}$ are the roots of f(x) = 0.

CAUTION

1. In an equation with rational coefficients, if $1+\sqrt[3]{2}$ is a root we cannot easily claim that $1-\sqrt[3]{2}$ is another root of the equation, since it involves a cubic surd.
2. In an equation with irrational coefficients, irrational roots may not occur in pairs. For example, the roots of the equation
   $x^2 - (2 + \sqrt{3})x + 2\sqrt{3} = 0$ are 2 and $\sqrt{3}$.

➤ **Example 7.** Solve the equation
$x^4 - 2x^2 - 22x^2 + 62x - 15 = 0$, having given that one root is $2 + \sqrt{3}$.

➤ ***Solution*** Since both $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are roots of the equation,

$$(x - 2 - \sqrt{3})(x - 2 + \sqrt{3}),$$

that is $x^2 - 4x + 1$, must be a factor of the left-hand side of the equation. Thus we have

$$(x^2 - 4x + 1)(x^2 + 2x - 15) = 0.$$

The roots are $2 \pm \sqrt{3}$, –5, 3.

➤ **Example 8.** Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is $\sqrt{5} + \sqrt{2}$.

➤ ***Solution*** Let $P(x) = x - (\sqrt{5} + \sqrt{2}) = [(x - \sqrt{5}) - \sqrt{2}]$

Now following the method used in the pervious example, using the conjugate, we get

$$P_1(x) = [(x - \sqrt{5}) - \sqrt{2}]\,[(x - \sqrt{5}) + \sqrt{2}]$$

$$= (x^2 - 2\sqrt{5}\,x + 5) - 2 = (x^2 + 3 - 2\sqrt{5}\,x)$$

$$P_2(x) = [(x^2 + 3) - 2\sqrt{5}\,x]\,[(x^2 + 3) + 2\sqrt{5}\,x]$$

$$= (x^2 + 3)^2 - 20x^2 = x^4 + 6x^2 + 9 - 20x^2$$

$$= x^4 - 14x^2 + 9$$

$$P(x) = ax^4 = 14ax^2 + 9a, \text{ where } a \in Q, a \neq 0.$$

The other zeroes of this polynomial are

$$\sqrt{5} - \sqrt{2}, -\sqrt{5} + \sqrt{2}, -\sqrt{5} - \sqrt{2}.$$

Rational root theory

If p and q are two numbers prime to each other and $\frac{p}{q}$ is a root of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, where $a_0, a_1, a_2, \ldots, a_n$ are all integers, then p is a factor of a_n and q is a factor of a_0.

Proof :

Since $\frac{p}{q}$ is a root,

$$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \ldots + a_{n-1}\frac{p}{q} + a_n = 0,$$

or, $a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \ldots + a_{n-1}pq^{n-1} + a_nq^n = 0$,

or, $a_0p^n = -q(a_1p^{n-1} + a_2qp^{n-2} + \ldots + a_n q^{n-1})$.

Both the factors on the right-hand side are integers. Therefore, q is a factor of a_0p^n. But q is prime to p, so it is a factor of a_0.

Similarly, since

$a_nq^n = -p(a_0p^{n-1} + a_1p^{n-2}q + ... + a_{n-1}q^{n-1})$,

We can prove that p is a factor of a_n. Thus the rational roots of an equation can be found by trials.

Corollary : Every rational root of the equation $x^n + a_1x^{n-1} + ... + a_n = 0$, where each $a_i (i = 1, 2, ..., n)$ is an integer, must be an integer. Moreover, every such root must be a divisor of the constant term a_n.

Proof :

Put $a_0 = 1$ in the preceding theorem. Since q must be a divisor of a_0, therefore we must have q = 1.

Hence every rational root must be an integer. Moreover, if p be such a root, then p must be a divisor of a_n.

➤ **Example 9.** Find the rational roots of

$6x^4 - x^3 + x^2 - 5x + 2 = 0.$

➤ ***Solution*** The factors of 6 are 1, 2, 3, 6 and those of 2 are 1 and 2. So the possible rational roots are ±1, ±2, $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$ of which $\frac{1}{2}$ and $\frac{2}{3}$ only satisfy the equation. Thus the roots are $\frac{1}{2}$ and $\frac{2}{3}$.

➤ **Example 10.** Find the integral roots of the equation

$x^4 - x^3 - 19x^2 + 49x - 30 = 0$

➤ ***Solution*** If α be any integral root, then α must be a divisor of the constant term – 30. Therefore the only integers which an possibly be the roots of the given equation are ±1, ±2, ±3, ±5, ±6, ±10, ±15 and 30.

By factor theorem we find that 1 and 2 are roots of the equation.

Thus $x^4 - x^3 - 19x^2 + 49x - 30 \equiv (x-1)(x-2)(x^2 + 2x - 15) \equiv (x-1)(x-2)(x-3)(x+5)$.

The given equation may, therefore, be written as

$(x-1)(x-2)(x-3)(x+5) = 0$

Hence the root are 1, 2, 3 and – 5.

➤ **Example 11.** Show that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + ... + \frac{L^2}{x-l} = x - m$$

where a, b, c,.....,l are real numbers, all different, cannot have any imaginary root.

➤ ***Solution*** If possible let α + iβ, β ≠ 0, be an imaginary root of the equation, then α – iβ is also a root.

$$\therefore \quad \frac{A^2}{\alpha+i\beta-a} + \frac{B^2}{\alpha+i\beta-b} + \frac{C^2}{\alpha+i\beta-c} + ... + \frac{L^2}{\alpha+i\beta-l} = \alpha + i\beta - m \; ..... \quad (1)$$

$$\text{and} \quad \frac{A^2}{\alpha-i\beta-a} + \frac{B^2}{\alpha-i\beta-b} + \frac{C^2}{\alpha-i\beta-c} + ... + \frac{L^2}{\alpha-i\beta-l} = \alpha - i\beta - m \; .... \quad (2)$$

Subtracting (1) from (2),

$$\left[\frac{A^2}{(\alpha-a)^2+\beta^2} + \frac{B^2}{(\alpha-b)^2+\beta^2} + ... + \frac{L^2}{(\alpha-l)^2+\beta^2} + 1\right] \times 2i\beta = 0.$$

This is possible only when β = 0. Hence the equation cannot have any imaginary root.

Practice Problems

1. One of the roots of $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ is $\sqrt{5} - 3$. Find the other roots.
2. Find the equation of the lowest degree with rational coefficients having $2 + \sqrt{3}$ and $\sqrt{5} - 2$ as two of its roots.
3. Two roots of the equation $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$ are $\alpha + i\beta$ and $\beta + i\alpha$. Find all the roots of the equation.

4. Solve the equation $x^4 + 2x^2 - 16x^2 - 22x + 7 = 0$, having given that $2 + \sqrt{3}$ is one root.

5. Find the biquadratic equation with rational coefficients one root of which is $\sqrt{3} - \sqrt{5}$.

6. Prove that the equation $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = 1$ has no imaginary roots.

7. Show that the roots of the equation

$\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} + \frac{4}{x-4} + \frac{5}{x-5} = 6$ are all real.

8. Find all the quadratic factors of $x^8 + x^4 + 1$ with real coefficients .

9. Find the rational roots of the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

10. Find the values of $p \in N$, so that the equation $x^3 - px + 1 = 0$ has atleast one rational root.

2.21 POLYNOMIAL EQUATION REDUCIBLE TO QUADRATIC EQUATION

I. An equation of the form

$(x-a)(x-b)(x-c)(x-d) = k$,

where $a < b < c < d$, $b - a = d - c$, can be solved by a change of variable.

i.e. $y = \frac{(x-a) + (x-b) + (x-c) + (x-d)}{4}$

$y = x - \frac{(a+b+c+d)}{4}$

➤ **Example 1**. Two solve

$$(x + a)(x + 2a)(x + 3a)(x + 4a) = \frac{9}{16}a^4.$$

Taking together the first and last of the factors on the left, and also the second and third, the equation becomes of the form we are now considering. We have

$$(x^2 + 5ax + 4a^2)(x^2 + 5ax + 6a^2) = \frac{9}{16}a^4.$$

Hence $(x^2 + 5ax)^2 + 10a^2(x^2 + 5ax) + 24a^2 = \frac{9}{16}a^4$,

$\therefore \quad x^2 + 5ax = -\frac{25}{4}a^2$, or else $x^2 + 5ax = -\frac{15}{4}a^2$.

Hence $\quad x + \frac{5}{2}a = 0$, or $x + \frac{5}{2}a = \pm\frac{a}{2}\sqrt{10}$.

Thus the roots are $-\frac{5}{2}a, -\frac{5}{2}a \pm \frac{a}{2}\sqrt{10}$.

Alternative :

Put $y = x + \frac{a + 2a + 3a + 4a}{4} \Rightarrow y = x + \frac{5a}{2}$

We get $\left(y - \frac{3a}{2}\right)\left(y - \frac{a}{2}\right)\left(y + \frac{a}{2}\right)\left(y + \frac{3a}{2}\right) = \frac{9}{16}a^4$

$\Rightarrow \quad \left(y^2 - \frac{9a^2}{2}\right)\left(y^2 - \frac{a^2}{2}\right) = \frac{9}{16}a^4$

We can solve this as a quadratic equation in y^2.

II. An equation of the form

$(x-a)(x-b)(x-c)(x-d) = kx^2$

where $ab = cd$, can be reduced to a collection of two quadratic equations by the substitution

$y = x + \frac{ab}{x}$.

➤ **Example 2.** Solve the equation

$$(x + 2)(x + 3)(x + 8)(x + 12) = 4x^2$$

➤ ***Solution*** Since $(-2)(-12) = (-3)(-8)$ we can write the equation as

$(x + 2)(x + 12)(x + 3)(x + 8) = 4x^2$ ...(i)

$\Rightarrow \quad (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$ ...(ii)

Check that $x = 0$ is not a root of (i).

Dividing by x^2 on both sides of (ii) we get

$\Rightarrow \quad \left(x + \frac{24}{x} + 14\right)\left(x + \frac{24}{x} + 11\right) = 4$ ...(iii)

Put $x + \frac{24}{x} = y$ then equation (iii) can be reduced to

$(y + 14)(y + 11) = 4$

or $y^2 + 25y + 150 = 0$

$\therefore$ $y_1 = -15$ and $y_2 = -10$

Thus the original equation is equivalent to the collection of equations :

$$\left[\begin{array}{l} x + \dfrac{24}{x} = -15 \\ x + \dfrac{24}{x} = -10 \end{array}\right.$$

i.e., $\left[\begin{array}{l} x^2 + 15x + 24 = 0 \\ x^2 + 10x + 24 = 0 \end{array}\right.$

Solving this collection, we get

$$x_1 = \frac{-15 - \sqrt{129}}{2},\ x_2 = \frac{-15 + \sqrt{129}}{2},\ x_3 = -6,$$

$x_4 = -4$.

III. An equation of the form $(x - a)^4 + (x - b)^4 = k$ can also be solved by a change of variable, i.e., making a substitution

$$y = \frac{(x - a) + (x - b)}{2}$$

➤ **Example 3.** Solve the equation

$(6 - x)^4 + (8 - x)^4 = 16$

➤ ***Solution*** After a change of variable

$$y = \frac{(6 - x) + (8 - x)}{2}$$

$\therefore$ $y = 7 - x$

or $x = 7 - y$

Now put $x = 7 - y$ in (i) we get

$(y - 1)^4 + (y + 1)^4 = 16$

$\therefore$ $y^4 + 6y^2 - 7 = 0$

$y^2 = -7$ gives imaginary roots

then $y_1 = -1$ and $y_2 = 1$

Thus $x_1 = 8$ and $x_2 = 6$ are the real roots.

IV. Reciprocal Equation

A reciprocal equation is one in which the coefficients are the same whether read in order backwards or forwards; or in which all the coefficients when read in order backwards differ in sign from the coefficients read in order forwards. Thus

$ax^3 + bx^2 + bx + a = 0$

$ax^4 + bx^3 + cx^2 + bx + a = 0$

$ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0$ are reciprocal equations.

To Solve $ax^4 + bx^3 + cx^2 + bx + a = 0$.

Divide by x^2. then we have

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0.$$

Now put $x + \dfrac{1}{x} = y$

then $x^2 + \dfrac{1}{x^2} = y^2 - 2$.

Hence $a(y^2 - 2) + by + c = 0$.

Let the two roots of the quadratic in y be α and β; then the roots of the original equation will be the four roots of the two equations

$$x + \frac{1}{x} = \alpha \text{ and } x + \frac{1}{x} = \beta.$$

To Solve $ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0$.

We have $a(x^5 - 1) + bx(x^3 - 1) + cx^2(x - 1) = 0$,

that is $(x - 1)\{a(x^4 + x^3 + x^2 + x + 1) + bx(x^2 + x + 1) + cx^2\} = 0$.

Hence $x = 1$, or else

$ax^4 + (b + a)x^2 + (a + b + c)x^2 + (b + a)x + a = 0$.

The last equation is a reciprocal equation of the fourth degree and is solved as before.

➤ **Example 4.** Solve the equation

$2x^4 + x^3 - 11x^2 + x + 2 = 0$ ...(1)

➤ ***Solution*** Since $x = 0$ is not a solution of the given equation. Dividing by x^2 in both sides of (1) we get

$$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0 \quad ...(2)$$

Putting $x + \dfrac{1}{x} = y$ in (2) we get $2(y^2 - 2) + y - 11 = 0$

$\Rightarrow$ $2y^2 + y - 15 = 0$

$\Rightarrow$ $\therefore$ $y_1 = -3$ and $y_2 = \dfrac{5}{2}$

Consequently, the original equation is equivalent to

$$\left\{\begin{array}{l} x + \dfrac{1}{x} = -3 \\ x + \dfrac{1}{x} = \dfrac{5}{2} \end{array}\right.$$

We find that

$$x_1 = \frac{-3-\sqrt{5}}{2}, x_2 = \frac{-3+\sqrt{5}}{2}, x_3 = \frac{1}{2}, x_4 = 2.$$

➤ **Example 5.** Solve
$6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0.$

➤ ***Solution*** One root is –1. Dividing by x + 1, we have $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$ Dividing by x^2 and grouping the terms,

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$$

If $z = x + \frac{1}{x}$, this becomes

$6(z^2 - 2) + 5z - 38 = 0$, that is $6z^2 + 5z - 50 = 0$;

$\therefore\quad z = \frac{5}{2}$ or $-\frac{10}{3}$, and x is given by $x + \frac{1}{x} = \frac{5}{2}$ or $-\frac{10}{3}$.

We find $x = 2, \frac{1}{2}, -3$ or $-\frac{1}{3}$. Thus the roots are

$$-1, 2, \frac{1}{2}, -3, -\frac{1}{3}.$$

V. Trinomial Equation

Equations of the form $ax^{2n} + bx^n + c = 0$, $a \neq 0$, $n \in N$
Substitute $x^n = t$

➤ **Example 6.** Solve the biquadratic equation
$x^4 - 5x^2 + 4 = 0$

➤ ***Solution*** Put $x^2 = t$
We get $t^2 - 5t + 4 = 0$
$\Rightarrow\ t = 1, 4$
$\Rightarrow\ x = \pm 1, \pm 2.$

➤ **Example 7.** Solve the biquadratic equation
$2x^4 + 2x^2 + 3 = 0$

➤ ***Solution*** We first reduce the equation to $x^4 + x^2 + \frac{3}{2} = 0$

$$x^4 + x^2 + \frac{3}{2} = \left(x^4 + 2\sqrt{\frac{3}{2}}x^2 + \frac{3}{2}\right) - 2\left(\sqrt{\frac{3}{2}} - 1\right)x^2$$

$$= \left(x^2 + \sqrt{\frac{3}{2}}\right)^2 - (\sqrt{6} - 1)x^2$$

$$= \left(x^2 + x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}}\right)\left(x^2 - x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}}\right)$$

The first equation $\quad x^2 + x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}} = 0$

has a negative discriminant :

$$D = (\sqrt{\sqrt{6} - 1})^2 - 4\sqrt{\frac{3}{2}} = -1 - \sqrt{6}$$

and, consequently, its roots

$$x_{1.2} = -\frac{\sqrt{\sqrt{6}-1}}{2} \pm i\frac{\sqrt{\sqrt{6}+1}}{2}$$

Similarly we find the roots of the second equation :

$$x^2 - x\sqrt{\sqrt{6} - 1} + \sqrt{\frac{3}{2}} = 0$$

They are

$$x_{3.4} = \frac{\sqrt{\sqrt{6}-1}}{2} \pm i\frac{\sqrt{\sqrt{6}+1}}{2}.$$

Practice Problems

Solve the following equations :

1. $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$
2. $2x^4 - x^3 - 11x^2 - x + 2 = 0$
3. $\frac{2x}{3x^2 - x + 2} - \frac{7x}{3x^2 + 5x + 2} = 1$
4. $(x - 1)^4 + (x - 5)^4 = 82$
5. $(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 5$
6. $x^2 + \frac{x^2}{(x + 1)^2} = 3$

2.22 RELATION BETWEEN ROOTS AND COEFFICIENTS

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ and $f(x) = 0$ have n roots $\alpha_1, \alpha_2, ..., \alpha_n$.

Then $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = a_0(x-\alpha_1)(x-\alpha_2) ..... (x-\alpha_n)$,

or, $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0}$

$= (x-\alpha_1)(x-\alpha_2) ... (x-\alpha_n)$,

or, $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0}$

$= x^n - \sum\alpha_1x^{n-1} + \sum\alpha_1\alpha_2x^{n-2} - \sum\alpha_1\alpha_2\alpha_3x^{n-3} + ... + (-1)^n\alpha_1\alpha_2...\alpha_n$.

Equating the coefficients of like powers of x on each side of the identity gives

$$\sum\alpha_1 = -\frac{a_1}{a_0},$$

$$\sum\alpha_1\alpha_2 = \frac{a_2}{a_0},$$

$$\sum\alpha_1\alpha_2\alpha_3 = \frac{a_3}{a_0},$$

....

$$\sum\alpha_1\alpha_2......\alpha_{n-1} = (-1)^{n-1}\frac{a_{n-1}}{a_0}.$$

$$\alpha_1\alpha_2......\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Therefore, the sum of the roots is equal to $-\frac{a_1}{a_0}$, the sum of their products in pair is equal to $\frac{a_2}{a_0}$ and so on.

For a **cubic equation** $ax^3 + bx^2 + cx + d = 0$, whose roots are α, β, γ

S_1 = Sum of the roots $= \Sigma\alpha = \alpha + \beta + \gamma = -\frac{b}{a}$

S_2 = Sum of the product of the roots taken two roots at a time $= \Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

S_3 = Product of the roots $= \alpha\beta\gamma = -\frac{d}{a}$

This can be verified by writing

$$ax^3 + bx^2 + cx + d \equiv a(x-\alpha)(x-\beta)(x-\gamma)$$

and comparing coefficients of like powers.

Similarly for a biquadratic equation

$ax^4 + bx^3 + cx^2 + dx + e = 0$ whose roots are $\alpha, \beta, \gamma, \delta$,

$S_1 = \Sigma\alpha = -\frac{b}{a}$ $S_2 = \Sigma\alpha\beta = \frac{c}{a}$

$S_3 = \Sigma\alpha\beta\gamma = -\frac{d}{a}$ $S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$

STUDY TIP It should be remarked that the sum of the roots of any polynomial equation will be zero provided the term one degree lower than the highest is absent.

➤ **Example 1.** Solve the equation

$$a^4\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^4\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^4\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^4.$$

➤ ***Solution*** The equation is clearly satisfied by x = a. by x = b, and by x = c. Also, since the coefficient of x^3 is zero, the sum of the roots is zero. Hence the remaining root must be – a – b – c.

➤ **Example 2.** Let α and β be the roots of the equation $x^3 + ax^2 + bx + c = 0$ satisfying the relation $\alpha\beta + 1 = 0$. Prove that $c^2 + ac + b + 1 = 0$.

➤ ***Solution*** If α, β and γ be the roots of the given equation, then we have

$\alpha + \beta + \gamma = -a$ ...(1)

$\alpha\beta + \alpha\gamma + \beta\gamma = b$ ...(2)

and $\alpha\beta\gamma = -c$ ...(3)

Also, we have $\alpha\beta + 1 = 0$ ...(4)

Putting $\alpha\beta = -1$ in equation (3), we have $\gamma = c$
Putting the value of γ in equation (1),
we have $\alpha + \beta = -a - c$
Now, putting the above values in equation (2), we have

$$\alpha\beta + \gamma(\alpha + \beta) = b$$

i.e. $-1 - c(a + c) = b$

i.e. $c^2 + ac + b + 1 = 0$

➤ **Example 3.** Let u, v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove or disprove uv is rational.

➤ ***Solution*** Let $x^3 + ax^2 + bx + c = 0$ be the cubic polynomial of which u, v and uv are the roots and a, b, c are all rationals.

$$u + v + uv = -a$$

$\Rightarrow$ $u + v = -a - uv$, $\quad uv + uv^2 + u^2v = b$

and $u^2v^2 = -c$

$$b = uv + uv^2 + u^2v = uv\,(1 + v + u)$$
$$= uv\,(1 - a - uv) \qquad = (1 - a)\,uv - u^2v^2$$
$$= (1 - a)\,uv + c$$

i.e., $uv = \dfrac{(b - c)}{1 - a}$ and since a, b, c are rational, uv is rational.

STUDY TIP When it is known that two or more roots of an equation are connected by any given relation, then it becomes easier to find these roots with the help of relations between roots and coefficients.

➤ **Example 4.** Solve completely the equation

$$x^4 - 5x^3 + 11x^2 - 13x + 6 = 0$$

using the fact that two of its roots α and β are connected by the relation $3\alpha + 2\beta = 7$.

➤ ***Solution*** From the given relation $\beta = \dfrac{7 - 3\alpha}{2}$.

Replacing x by $\dfrac{7 - 3x}{2}$ in $f(x) = x^4 - 5x^3 + 11x^2 - 13x + 6$, we get

$$\left(\frac{7-3x}{2}\right)^4 - 5\left(\frac{7-3x}{2}\right)^3 + 11\left(\frac{7-3x}{2}\right)^2 - 13\frac{7-3x}{2} + 6$$

Let it reduce to

$$\phi(x) = 81x^4 - 486x^3 + 1152x^2 - 1242x + 495.$$

The H.C.D. of f(x) and $\phi(x)$ is $x - 1$.

$$\therefore \quad \alpha = 1 \text{ and } \quad \beta = \frac{7 - 3}{2} = 2.$$

Let the other roots of $f(x) = 0$ by γ and δ.
By the relation between roots and coefficients,

$1 + 2 + \gamma + \delta = 5$, $\quad$ or, $\quad \gamma + \delta = 2$

and $1 \,.\, 2 \,.\, \gamma \,.\, \delta = 6$, $\quad$ or, $\quad \gamma\delta = 3$.

Solving a quadratic equation we get $\gamma = 1 + i\sqrt{2}$ and $\delta = 1 - i\sqrt{2}$.

Hence the roots are $1, 2, 1 \pm i\sqrt{2}$.

➤ **Example 5.** Solve the cubic equation $9x^3 - 27x^2 + 26x - 8 = 0$, given that one of the roots of this equation is double to other.

➤ ***Solution*** Let the roots be α, 2α and β.

Now, $3\alpha + \beta = -\dfrac{-27}{9} = 3$

$$\Rightarrow \quad \beta = 3(1 - \alpha) \qquad ...(1)$$

$$2\alpha^2 + 3\alpha\beta = \frac{26}{9} \qquad ...(2)$$

$$2\alpha^2\beta = +\frac{8}{9} \qquad ...(3)$$

From Eq. (1) and Eq. (2), we get

$$2\alpha^2 + 3\alpha \times 3(1 - \alpha) = \frac{26}{9}$$

$\Rightarrow$ $63\alpha^2 - 81\alpha + 26 = 0$

$\Rightarrow$ $(21\alpha - 13)(3\alpha - 2) = 0$

So $\alpha = \dfrac{13}{21}$ or $\dfrac{2}{3}$

If $a = \dfrac{13}{21}$ $\quad \therefore \quad \beta = 3\left(1 - \dfrac{13}{21}\right) = \dfrac{24}{21} = \dfrac{8}{7}$

This leads to $2\alpha^2\beta = 2 \times \dfrac{169}{441} \times \dfrac{8}{7} \neq \dfrac{8}{9}$.

(a contradiction)

So taking $a = \frac{2}{3}$,

$\therefore \quad \beta = 3\left(1 - \frac{2}{3}\right) = 3 \times \frac{1}{3} = 1$

Hence $\quad \alpha + 2\alpha + \beta = \frac{2}{3} + \frac{4}{3} + 1 = 3,$

$$2\alpha^2 + 3\alpha\beta = 2 \times \frac{4}{9} + \frac{3 \times 2}{3} \times 1 = \frac{26}{9},$$

and $2\alpha^2\beta = 2 \times \frac{4}{9} \times 1 = \frac{8}{9}$

Thus, the roots are $\frac{2}{3}, \frac{4}{3}$ and 1.

➤ **Example 6.** Show that all the roots of the equation $ax^3 + x^2 + x + 1 = 0$ cannot be real, where $a \in R$.

➤ ***Solution*** If a = 0 then all roots are obviously non-real therefore we have $a \neq 0$.

Let the roots be x_1, x_2, x_3 then we have

$x_1 + x_2 + x_3 = -\frac{1}{a}$, $x_1x_2 + x_2x_3 + x_2x_1 = \frac{1}{a}$, $x_1x_2x_3 = -\frac{1}{a}$

Now, if possible, let all the roots be real. Consider the expression

$x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 = (x_1 x_2 + x_2 x_3 + x_3 x_1)^2 - 2x_1 x_2 x_3 (x_1 + x_2 + x_3)$

$$= \frac{1}{a^2} - 2\left(-\frac{1}{a}\right)\left(-\frac{1}{a}\right) = -\frac{1}{a^2} < 0$$

which is contradictory

since $x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2$ must be essentially non-negative if all the roots are real.

Practice Problems

1. Solve the equation $2x^3 + x^2 - 7x - 6 = 0$, given that the difference between two of the roots is 3.
2. Solve the equation $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, the roots being in A.P.
3. Solve $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.
4. Solve $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of 3 : 4.
5. Solve $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal but opposite in sign.
6. Solve $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of the roots being 2.
7. If one root of the equation $x^3 + 2ax^2 - b = 0$, is equal to the sum of the other two, then show that $a^3 = b$.
8. Solve $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in G.P.

2.23 TRANSFORMATION OF POLYNOMIAL EQUATION

Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x) = 0$, and suppose that we require the equation whose roots are $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$,.... where $\phi(x)$ is a given function of x.

Let $y = \phi(x)$ and suppose that from this equation we can find x as a single-valued function of y, which we denote by $\phi^{-1}(y)$.

Transforming the equation $f(x) = 0$ by the substitution $x = \phi^{-1}(y)$, we obtain $f\{\phi^{-1}(y)\} = 0$, which is the equation required.

A case in which x is not a single-valued function of y is also shown below.

STUDY TIP The following transformations are often required.

Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x) = 0$, then

(1) the equation whose roots are $-\alpha, -\beta, -\gamma, \ldots$ if $f(-x) = 0$;

(2) the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma$, is $f(1/x) = 0$

(3) the equation whose roots are $k\alpha, k\beta, k\gamma, \ldots$ is $f(x/k) = 0$.

(4) The equation whose roots are $\alpha - h, \beta - h, \gamma - h$ is $f(x + h) = 0$.

➤ **Example 1.** Solve $6x^3 - 11x^2 + 6x - 1 = 0$ if roots of the equation are in H.P.

➤ ***Solution*** Putting $x = \frac{1}{y}$ in the given equation we get

$$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0 \Rightarrow 6 - 11 + 6y^2 - y^3 = 0$$

$\Rightarrow \quad y^3 - 6y^2 + 11y - 6 = 0$ ...(1)

Now roots of (1) are in A.P.

Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$

Then sum of roots $\alpha - \beta + \alpha + \alpha + \beta = 6$

$\Rightarrow \quad 3\alpha = 6 \quad \therefore \quad \alpha = 2$

Product of roots $(\alpha - \beta)\,\alpha\,(\alpha + \beta) = 6$

$\therefore \quad 2(4 - \beta^2) = 6 \therefore \quad \beta = \pm 1$

$\therefore$ Roots of (1) are 1, 2, 3

Hence roots of the given equation are $1, \frac{1}{2}, \frac{1}{3}$

➤ **Example 2.** If a, b, c be the roots of equation $x^3 + px^2 + qx + r = 0$, then find a cubic equation whose roots are $a(b + c), b(c + a), c(a + b)$.

➤ ***Solution*** Given $a + b + c = -p$,

$ab + bc + ca = q, \quad abc = -r$

Let $x = a(b + c) = ab + ac = ab + bc + ca - bc = q - bc = q + \frac{r}{a}$

$\Rightarrow \quad \frac{r}{a} = x - q \Rightarrow \quad a = \frac{r}{x - q}$ ...(1)

Since a is a root of given equation so $a^3 + pa^2 + qa + r = 0$

Put a from (1)

$$\Rightarrow \quad \frac{r^3}{(x-q)^3} + p\frac{r^2}{(x-q)^2} + q\frac{r}{(x-q)} + r = 0$$

$$\Rightarrow \quad \frac{r^2}{(x-q)^3}$$

$$r^2 + pr(x - q) + x(x - q)^2 = 0$$

➤ **Example 3.** If α, β, γ be the roots of the equation $x^3 - px^2 + r = 0$, find a cubic equation whose roots are

$$\frac{\beta+\gamma}{\alpha}, \frac{\gamma+\alpha}{\beta}, \frac{\alpha+\beta}{\gamma}.$$

➤ ***Solution*** Let

$$y = \frac{\beta+\gamma+\alpha-\alpha}{\alpha} = \frac{p-\alpha}{\alpha} = \frac{p}{\alpha} - 1 \quad \left(\because \sum\alpha = p\right),$$

or, $\quad \alpha = \frac{p}{y+1} \quad \therefore \quad x = \frac{p}{y+1}$.

Replacing x by $\frac{p}{y+1}$ in the equation, we have

$$\frac{p^3}{(y+1)^3} - \frac{p^3}{(y+1)^2} + r = 0,$$

or, $\quad r(y + 1)^3 - p^3(y + 1) + p^3 = 0$,

or, $\quad ry^3 + 3ry^2 + (3r - p^3)\,y + r = 0$.

It is the required equation in y.

➤ **Example 4.** If α, β, γ are the roots of the cubic $x^3 + x + 2 = 0$, find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

➤ ***Solution*** Since α, β, γ are the roots of the cubic

$x^3 + x + 2 = 0$ ...(1)

then $\Sigma\alpha = 0, \quad \Sigma\alpha\beta = 1$,

$\alpha\beta\gamma = -2$ ...(2)

If y is a root of the required equation, then

$$y = (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = (\alpha+\beta+\gamma-\gamma)^2 - \frac{4\alpha\beta\gamma}{\gamma}$$

$$= (0 - \gamma)^2 + \frac{8}{\gamma} \qquad \{\text{from (2)}\}$$

$\Rightarrow \quad y = \gamma^2 + \frac{8}{\gamma}$ [replacing γ by x which is root of (1)]

$\therefore \quad y = x^2 + \frac{8}{\gamma}$ or $x^3 - yx + 8 = 0$ ...(3)

The required equation is obtained by eliminating x between (1) and (3).

Now subtracting (1) from (3), we get

$$(1 + y)\,x - 6 = 0 \qquad \text{or} \qquad x = \frac{6}{1+y}$$

substituting in (1), we get

$$\left(\frac{6}{1+y}\right)^3 + \left(\frac{6}{1+y}\right) + 2 = 0$$

or $\quad y^3 + 6y^2 + 9y + 112 = 0$
which is the required equation.

➤ **Example 5.** If α is a root of the equation
$x^4 + px^3 - 6x^2 - px + 1 = 0,$
then show that $\frac{1+\alpha}{1-\alpha}$ is also a root. Hence show that the other two roots are $-\left(\frac{1}{\alpha}\right), \frac{(\alpha-1)}{(\alpha+1)}$.

➤ ***Solution*** Putting $x = \frac{1+\alpha}{1-\alpha}$ in the given equation, we get

$$\left(\frac{1+\alpha}{1-\alpha}\right)^4 + \left(\frac{1+\alpha}{1-\alpha}\right)^3 - 6\left(\frac{1+\alpha}{1-\alpha}\right)^2 - p\left(\frac{1+\alpha}{1-\alpha}\right) + 1 = 0$$

$\Rightarrow \quad [(1+\alpha)^4 + (1-\alpha)^4] + p[(1+\alpha)^3(1-\alpha) + (1+\alpha)^3] - 6(1+\alpha)^2(1-\alpha)^2 = 0$

$\Rightarrow \quad 2(1 + 6\alpha^2 + \alpha^4) + 4\alpha(1-\alpha)^2 p - 6(1 - 2\alpha^2 + \alpha^4) = 0$

$\Rightarrow \quad \alpha^4 + p\alpha^3 - 6\alpha^2 - p\alpha + 1 = 0$

which shows that $\frac{1+\alpha}{1-\alpha}$ is also a root.

Now replacing x by $-\frac{1}{x}$, in the given equation we note that the equation does not change. Hence if α is a root then $-\frac{1}{\alpha}$ is also a root.

Now since $-\frac{1}{\alpha}$ is a root $\dfrac{1+\left(-\frac{1}{\alpha}\right)}{1-\left(-\frac{1}{\alpha}\right)}$ is a root from above.

Thus the roots are $\alpha, -\frac{1}{\alpha}, \frac{1+\alpha}{1-\alpha}$ and $\frac{(\alpha-1)}{(\alpha+1)}$.

Practice Problems

1. Find the equation each of the whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.
2. Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$, the roots of which are in H.P.
3. If α, β, γ are the roots of $x^3 + x^2 - 4x + 7 = 0$, find the reduced cubic equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$.
4. Find the equation whose roots are those of the equation $6x^3 - 5x^2 - \frac{1}{4} = 0$, each multiplied by c, and find the least value of c in order that the resulting equation may have integral coefficients with unity for the coefficient of the highest power.
5. If a, b, c are the roots of the equation $x^3 - px^2 + qx - r = 0$ find the value of
   (a) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ (b) $\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}$
6. If a, b, c are the roots of $x^3 + qx + r = 0$, find the value of
   (a) $(b-c)^2 + (c-a)^2 + (a-b)^2$
   (b) $\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}$

2.24 SYMMETRIC FUNCTIONS OF ROOTS

Symmetrical expressions

An expression which is unaltered by interchanging any pair of the letters which it contains is said to be a symmetrical expression. Thus $a + b + c$, $bc + ca + ab$, $a^3 + b^3 + c^3 - 3abc$ are symmetrical expressions.
Expressions which are unaltered by a cyclical change of the letters involved in them are called cyclically

symmetrical expressions. For example, the expression (b – c) (c – a) (a – b) is a cyclically symmetrical expression since it is unaltered by changing a into b, b into c, and c into a.

It is clear that the product, or the quotient, of two symmetrical expressions is symmetrical, for if neither of two expressions is altered by an interchange of two letters, their product, or their quotient, cannot be altered by such interchange.

It is also clear that the product, or the quotient, of two cyclically symmetrical expressions is cyclically symmetrical.

Every rational symmetric function of the roots of a polynomial equation is expressible in terms of the elementary symmetric functions of the same and therefore in terms of the coefficients of the equation. Without knowing the separate values of the roots in terms of the coefficients, we can calculate the values of symmetric functions of roots in terms of the coefficients. For example, if α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$, we can find out the value of

(i) $\alpha\beta + \beta\gamma + \gamma\alpha$,

(ii) $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$,

(iii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

in terms of p, q and r and each of these relations are symmetric functions of the roots, since there will be no change in them if two of α, β, γ are interchanged. To find out the value of the above symmetric functions, we take the help of the following relations,

$$\alpha + \beta + \gamma = -p, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = q,$$
$$\alpha\beta\gamma = -r.$$

Now $\sum \alpha^2\beta = \sum \alpha \sum \alpha\beta - 3\alpha\beta\gamma = -p \times q - 3.(-r)$
$= 3r - pq$ and

$$\sum \alpha^2\beta\gamma = \alpha\beta\gamma \sum \alpha = -r \times (-p) = pr.$$

Important Formulae

(1) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
$$= a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

(2) $a^2 + b^2 + c^2 - ab - bc - ca$
$$= \frac{1}{2}\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

(3) $a^3 + b^3 + c^3 - 3abc$
$$= (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$$

Newton's Theorem

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of
$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$$
and if $S_r = \sum \alpha_1^r$, then

(i) $S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + S_1p_{r-1} + rp_r = 0$ when $r < n$

and

(ii) $S_r + p_1S_{r-1} + p_2S_{r-2} + \ldots + p_{n-1}S_{r-n+1} + p_nS_{r-n} = 0$ when $r \geq n$.

➤ **Example 1.** Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$ then find the value of $a^3 + b^3 + c^3$.

➤ ***Solution*** Let t be the root of the given cubic where t can take values a, b, c

hence $t^3 + t^2 - 333t - 1002 = 0$

or $t^3 = 1002 + 333t - t^2$

$$\therefore \quad \sum t^3 = \sum 1002 + 333\sum t - \sum t^2$$
$$= 3006 + 333\sum t - \left[\left(\sum t\right)^2 - 2\sum t_1t_2\right]$$

but $\sum t = -1$; $\sum t_1t_2 = -333$

$\therefore$ $a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666]$
$= 3006 - 333 - 667 = 3006 - 1000 = 2006$

➤ **Example 2.** If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + x = 0$,

show that $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2)$
$= (1 - q + s)^2 + (p - r)^2$.

➤ ***Solution*** Since $\alpha, \beta, \gamma, \delta$ are the roots of the equation
$$x^4 + px^3 + qr^2 + rx + x = 0,$$
therefore $x^4 + px^3 + qx^2 + rx + x$
$$\equiv (x - \alpha)(x - \beta)(x - \gamma)(x - \delta),$$
Substituting $x = i, -i$ successively, we have
$$(1 - q + s) - i(p - r) = (i - \alpha)(i - \beta)(i - \gamma)(i - \delta) \quad \ldots(1)$$
$$(1 - q + s) + i(p - r) = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \delta) \quad \ldots(2)$$
Multiplying corresponding sides of (1) and (2), we have

$$(1-q+s)^2+(p-r)^2=(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2).$$

➤ **Example 3.** If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, then prove that

$$\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} \times \frac{\alpha^2+\beta^2+\gamma^2}{2}$$

➤ ***Solution*** Since α, β, γ are the roots of the equation

$$x^3 + qx + r = 0, \qquad ...(1)$$

therefore $$\left.\begin{aligned}\alpha^3 + q\alpha + r &= 0,\\ \beta^3 + q\beta + r &= 0,\\ \gamma^3 + q\gamma + r &= 0\end{aligned}\right\} \qquad ...(2)$$

Adding the corresponding sides of (2), we have

$$\Sigma\alpha^3 + q\Sigma\alpha + 3r = 0$$

Since $\Sigma a = 0$, this yields,

$$\Sigma\alpha^3 = -3r \qquad ...(3)$$

Also, $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = -2q$ ...(4)

Multiplying (1) throughout by x^2, we find that α, β, γ are three of the roots of the equation

$$x^5 + qx^3 + rx^2 = 0 \qquad ...(5)$$

Substituting $x = \alpha, \beta, \gamma$ successively in (v) and adding, we get

$$\Sigma\alpha^5 + q\Sigma\alpha^3 + r\Sigma\alpha^2 = 0 \qquad ...(6)$$

Substituting the values of $\Sigma\alpha^3$ and $\Sigma\alpha^2$ from (3) and (4), we get

$$\Sigma\alpha^5 = 5qr$$

From (3), (4) and (7), we get

$$\frac{1}{5}\Sigma\alpha^5 = \left(\frac{1}{3}\Sigma\alpha^3\right)\left(\frac{1}{2}\Sigma\alpha^3\right)$$

2.25 TRIGONOMETRICAL METHOD OF SOLVING CUBIC EQUATION

We consider the trigonometrical identity

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta,$$

or, $$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0, \qquad ...(1)$$

Writting $z = r\cos\theta, r > 0$, in $z^3 + 3Hz + G = 0$, we have

$$\cos^3\theta + \frac{3H}{r^2}\cos\theta + \frac{G}{r^3} = 0. \qquad ...(2)$$

If we assume that the equations (1) and (2) are identical then comparing them,

$$\frac{3H}{r^2} = -\frac{3}{4} \text{ and } \frac{G}{r^3} = -\frac{1}{4}\cos 3\theta, \text{ or}$$

$$r = 2\sqrt{-H}, \cos 3\theta = \frac{-4G}{r^3} = \frac{-G}{2\sqrt{-H^3}} \qquad ...(3)$$

From (3), θ is known and then $z = 2\sqrt{-H}\cos\theta$ is obtained.

If α is any value of θ satisfying $\cos 3\theta = \dfrac{-G}{2\sqrt{-H^3}}$,

then $3\theta = 2n\pi \pm 3\alpha$. The three different values of $\cos\theta$ are $\cos\alpha$, $\cos\left(\frac{2\pi}{3}+\alpha\right)$ and $\cos\left(\frac{2\pi}{3}-\alpha\right)$

and hence the solutions are

$$2\sqrt{-H}\cos\alpha,\ 2\sqrt{-H}\cos\left(\frac{2\pi}{3}+\alpha\right)$$

and $$2\sqrt{-H}\cos\left(\frac{2\pi}{3}-\alpha\right).$$

If θ is real, $|\cos 3\theta| \le 1$.

$$\therefore \quad \left|\frac{G}{2\sqrt{-H^3}}\right| \le 1. \qquad \text{i.e.,} \qquad G^2 + 4H^3 \le 0.$$

In this case all the roots are real. Further in order that r is real, $H < 0$.

Thus the trigonometrical method to solve a cubic equation is generally applicable when the roots are real.

➤ **Example 1.** Solve $x^3 - 27x + 27 = 0$.

➤ ***Solution*** Here $H = -9$, $G = 27$, $G^2 + 4H^3 = 27^2 - 4.9^3 = 9^3 - 4.9^3 = -3.9^3 < 0$.

Therefore all the roots are real.

From above, $\cos 3\theta = \dfrac{-G}{2\sqrt{-H^3}} = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$.

$\therefore \quad 3\theta = 2n\pi + \frac{2\pi}{3}.$

Again $x = 2\sqrt{-H}\cos\theta = 6\cos\theta$.

The three different values of $\cos\theta$ are $\cos\frac{2\pi}{9}$, $\cos\frac{8\pi}{9}$ and $\cos\frac{4\pi}{9}$.

Hence the solutions are, $6\cos\frac{2\pi}{9}$, $6\cos\frac{8\pi}{9}$ and $6\cos\frac{4\pi}{9}$.

➤ **Example 2.** Show that the roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$ and hence find the equation whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

➤ ***Solution*** Let $7\theta = (2n+1)\pi$ where n is any integer.

$\therefore \quad \theta = \frac{2n+1}{7}\pi$ and $\cos\theta = \cos\frac{2n+1}{7}\pi$

where n = 0, 1,....., 6.

Explicitly $\cos\theta = \cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$, $\cos\pi$, $\cos\frac{9\pi}{7}$ $\cos\frac{11\pi}{7}$, $\cos\frac{13\pi}{7}$.

Again $\cos\frac{13\pi}{7} = \cos\frac{\pi}{7}$, $\cos\frac{11\pi}{7} = \cos\frac{3\pi}{7}$, $\cos\frac{9\pi}{7} = \cos\frac{5\pi}{7}$.

Therefore the distinct values of $\cos\theta$ are

$\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ and -1 (since $\cos\pi = -1$)

Now $7\theta = (2n+1)\pi$, or, $4\theta = (2n+1)\pi - 3\theta$.

$\therefore \quad \cos 4\theta = -\cos 3\theta$,

or, $\quad 2\cos^2 2\theta - 1 = -4\cos^3\theta + 3\cos\theta$,

or, $\quad 2(2\cos^2\theta - 1)^2 - 1 = -4\cos^3\theta + 3\cos\theta$,

or, $\quad 8\cos^4\theta - 8\cos^2\theta + 1 = -4\cos^3\theta + 3\cos\theta$,

or, $\quad 8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$.

Putting $\cos\theta = x$, we get $8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$.

The roots of this equation in x are -1, $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$.

For the root -1, $x + 1$ is a factor of $8x^4 + 4x^3 - 8x^2 - 3x + 1$. Cancelling this factor we get

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

This is the required equation.

Transforming the equation by $y = \frac{1}{x^2}$, we get the equation

whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

We have $\quad 8x^3 - 4x = 4x^2 - 1$,

or, $\quad 4x(2x^2 - 1) = 4x^2 - 1$,

or, $\quad 4x\left(\frac{2}{y} - 1\right) = \frac{4}{y} - 1$,

or, $\quad 4x(2 - y) = 4 - y$.

Squaring, $\quad 16x^2(2 - y)^2 = (4 - y)^2$,

or, $\quad \frac{16}{y}(4 - 4y + y^2) = 16 - 8y + y^2$,

or, $\quad y^3 - 24y^2 + 80y - 64 = 0$.

It is the equation in y whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

Practice Problems

1. If α, β, γ be the roots of the $ax^3 + bx^2 + cx + d = 0$ then find the values of ,

(i) $\sum \alpha^2$ (ii) $\sum \frac{1}{\alpha}$ (iii) $\sum \alpha^2\beta$

2. If a, b, c be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the values of

(i) $(b + c - 3a)(c + a - 3b)(a + b - 3c)$.

(ii) $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$.

(iii) $\left(\frac{1}{a^2}-\frac{1}{bc}\right)\left(\frac{1}{b^2}-\frac{1}{ca}\right)\left(\frac{1}{c^2}-\frac{1}{ab}\right)$.

3. Find the roots of the equation $x^3 - 3x + 1 = 0$.

4. Solve the equation $8x^3 - 36x + 27 = 0$

2.26 COMMON ROOTS IN POLYNOMIAL EQUATIONS

If $\psi(x)$ is the highest common factor (H.C.F.) of f(x) and g(x), then the roots of $\psi(x) = 0$ are the common roots of f(x) = 0 and g(x) = 0.

➤ **Example 1.** Find the common roots of $x^4 + 5x^3 - 2x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ and hence solve the equations.

➤ ***Solution*** We find that $4(x^2 - 5x + 6)$ is H.C.F. of the two equations

and hence, the common roots are the roots of

$x^2 - 5x + 6 = 0$ i.e., x = 3 or x = 2.

Now, $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ ....(1)

and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ ....(2)

have 2 and 3 as their common roots.

➤ **Example 2.** Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$). Find the sum of the squares of the roots of the cubic polynomial.

➤ ***Solution*** Since cubic is divisible by both

$x^2 + ax + b$ and $x^2 + bx + a$

$\therefore$ $x^2 + ax + b$

and $x^2 + bx + a$ must have a common roots.

$x^2 + ax + b = 0$ $\quad - x^2 + bx + a = 0$

subtract

$\Rightarrow$ $x(a - b) = (a - b)$

$x = 1$ $\quad \therefore$ common root is 1

$x^2 + ax + b = 0$ — roots 1, α

$1 \cdot \alpha = b \quad \Rightarrow \quad \alpha = b$

$x^2 + bx + a = 0$ — roots 1, β

$1 \cdot \beta = a \quad \Rightarrow \quad \beta = a$

$\Rightarrow$ roots of cubic be 1, a, b

product of the roots be

$1 \cdot a \cdot b = -72$ ....(1)

and $a + b + 1 = 0$ ....(2)

(from $x^2 + ax + b = 0$ put x = 1)

$\therefore$ $a - \frac{72}{b} = -1$

$\Rightarrow$ $a^2 + a - 72 = 0$ $\quad (a + 9)(a - 8) = 0$

$a = -9, 8$ $\quad \therefore$ roots are $1, -9, 8$

$\Rightarrow$ sum of their squares = 1 + 81 + 64 = 146

Method of Substraction

If α is a common root of f(x) = 0 and g(x) = 0 then α is also a common root of f(x) – g(x) = 0 ; but not all roots f(x) – g(x) = 0 are common roots of f(x) and g(x).

Proof :

Let α be a common root

Then $f(\alpha) = 0$ and $g(\alpha) = 0$ ...(i)

The equation f(x) – g(x) = 0 is satisfied by $x = \alpha$ because of (i)

However, consider a number β such that

$f(\beta) = g(\beta) = k \neq 0$ ...(ii)

Then β satisfies the equation f(x) – g(x) = 0 because of (ii) but we know that β is not a common root of f(x) and g(x).

Hence all roots of f(x) – g(x) = 0 are not necessarily common roots. Here we solve f(x) – g(x) = 0. Only those roots of $f(x) - g(x) = 0$ which satisfy $f(x) = 0$ (or $g(x) = 0$) are the common roots.

➤ **Example 3.** Find the common roots of the equations

$$x^3 - 2x^2 - x + 2 = 0 \text{ and } x^3 + 6x^2 + 11x + 6 = 0.$$

➤ ***Solution*** Substract the two equations

We get $\quad 8x^2 + 12x + 4 = 0$

$$2x^2 + 3x + 1 = 0 \quad \Rightarrow \quad x = -1, -\frac{1}{2}$$

Verify $x = -1, -\frac{1}{2}$ in one of the equations, say

$$x^3 - 2x^2 - x + 2 = 0.$$

Only $x = -1$ satisfies this equation.

It means that $x = -1$ is the only common root.

➤ **Example 4.** Find the values of a so that the following equations have common root(s) :

$$x^3 - 2x^2 - x + a = 0, \quad x^2 - 5x + 2a = 0$$

➤ ***Solution*** $x^3 - 2x^2 - x + a = 0$ ...(i)

$$x^2 - 5x + 2a = 0 \qquad \text{...(ii)}$$

$x = 0$ is a common root when $a = 0$.

Now multiply (ii) by x and substract from (i).

So as to cancel the x^3 term.

$$-3x^2 + (1 + 2a)x - a = 0 \qquad \text{...(iii)}$$

Multiply (ii) by 3 and add to (iii)

$$(2a - 14)x + 5a = 0$$

$$\Rightarrow \quad x = \frac{5a}{14 - 2a}$$

For this to be the common root we put this root into (ii).

$$\left(\frac{5a}{14 - 2a}\right)^2 - 5\left(\frac{5a}{14 - 2a}\right) + 2a = 0$$

$$\Rightarrow \quad a[25a - 25(14 - 2a) + 2(14 - 2a)^2] = 0$$

$$\Rightarrow \quad a(8a^2 - 37a + 42) = 0$$

$$\Rightarrow \quad a(a - 2)(8a - 21) = 0$$

$$\Rightarrow \quad a = 0, 2, \frac{21}{8}.$$

The value of a for which we get common roots are

$$a = 0, 2, \frac{21}{8}.$$

2.27 MULTIPLE ROOTS

If f(x) contains a factor $(x - \alpha)^r$, then $f'(x)$ contains a factor $(x - \alpha)^{r-1}$. Therefore if f(x) and $f'(x)$ have no common factor, no factor in f(x) will be repeated ; hence the equation $f(x) = 0$ has or has not equal roots, according as f(x) and $f'(x)$ have or have not a common factor involving x.

Proof :

If α be an r-multiple root of $f(x) = 0$, whose degree is n, then

$$f(x) = (x - \alpha)^r \phi(x)$$

where $\phi(x)$ is a polynomial in x of degree $n - r$ and it is not divisible by $x - \alpha$.

Now $f'(x) = r(x - \alpha)^{r-1}\phi(x) + (x - \alpha)^r\phi'(x)$

$$= (x - \alpha)^{r-1}\{r\phi(x) + (x - \alpha)\phi'(x)\}$$

$$= (x - \alpha)^{r-1}\psi(x), \text{ where } \psi(x)$$

$$= r\phi(x) + (x - \alpha)\phi'(x).$$

$\psi(x)$ is not divisible by $(x - \alpha)$. Therefore, α is an $(r - 1)$multiple root of $f'(x) = 0$. Consequently $(x - \alpha)^{r-1}$ is the highest common divisor of f(x) and $f'(x)$.

If f(x) has no other multiple root, $(x - \alpha)^{r-1}$ is the highest common divisors of f(x) and $f'(x)$. So, to determine the multiple roots of $f(x) = 0$, we find out the H.C.F. of f(x) and $f'(x)$ and if it is of the form $(x - \alpha)^{p-1}(x - \beta)^{q-1}$...., then α, β,.... are the multiple roots of $f(x) = 0$ with multiplicities p, q,...... respectively.

Corollary: If α is an r-multiple root of $f(x) = 0$, it is then an $(r - 1)$ multiple root of $f'(x) = 0$, an $(r - 2)$ multiple root of $f''(x) = 0$, and so on.

➤ **Example 1**. Find the condition that the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may have two roots equal.

In this case the equations $f(x) = 0$, and $f'(x) = 0$, that is

$$ax^3 + 3bx^2 + 3cx + d = 0 \qquad \text{...(i)}$$

$$ax^2 + 2bx + c = 0 \qquad \text{...(ii)}$$

must have a common root, and the condition required will be obtained by eliminating x between these two equations.

By combining (i) and (ii), we have

$$bx^2 + 2cx + d = 0 \qquad \text{...(iii)}$$

From (ii) and (iii), we obtain

$$\frac{x^2}{2(bd-c^2)} = \frac{x}{bc-ad} = \frac{1}{2(ac-b^2)};$$

thus the required conditions is

$$(bc-ad)^2 = 4(ac-b^2)(bd-c^2).$$

➤ **Example 2.** Find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$ if two roots are equal.

➤ ***Solution*** Let roots be α, α and β

$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4}$

$\Rightarrow \quad 2\alpha + \beta = -5$ ....(i)

$\therefore \quad \alpha . \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4}$

$\Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ & $\alpha^2\beta = \frac{6}{4}$

from equation (i) $\alpha^2 + 2\alpha(-5-2\alpha) = -\frac{23}{4}$

$\Rightarrow \quad \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4}$

$\Rightarrow \quad 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \quad \alpha = 1/2, -\frac{23}{6}$ when $\alpha = \frac{1}{2}$

from equation (i) $\alpha^2\beta = \frac{1}{4}(-5-1) = -\frac{3}{2}$

when $\alpha = \frac{23}{6}$

$$\alpha^2\beta = \frac{23 \times 23}{36}\left(-5-5x\left(-\frac{23}{6}\right)\right) \neq \frac{3}{2}$$

$\Rightarrow \quad \alpha = \frac{1}{2}, \quad \beta = -6$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$

Practice Problems

1. The equations $2x^3 + 5x^2 - 6x - 9 = 0$ and $3x^3 + 7x^2 + 7x^2 - 11x - 15 = 0$ have two common roots. Find them.
2. Determine the common roots of the equation $6x^3 + 7x^2 - x - 2 = 0$, and $6x^4 + 19x^3 + 17x^2 - 2x - 6 = 0$
3. If $x^3 + px^2 + qx + r = 0$ and $x^3 + ax^2 + bx + c = 0$ have two common roots, find the quadratic equation having these common roots as roots.
4. Find the roots common to the equations $x^5 - x^3 + x^2 - 1 = 0$, $x^4 = 1$.
5. Show that $x^3 - 2x^2 - 2x + 1 = 0$ and $x^4 - 7x^2 + 1 = 0$ have two roots in common.
6. Find the solution of the following equations which have common roots : $2x^4 - 2x^3 + x^2 + 3x - 6 = 0$, $4x^4 - 2x^3 + 3x - 9 = 0$
7. Find the ratio of b to a in order that the equations $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ may have (1) one, (2) two roots in common.
8. Find the condition that $ax^3 + bx + c$ and $a'x^3 + b'x + c'$ may have a common linear factor.
9. If the two equations $ax^3 + 3bx^2 + 3cx + d = 0$ and $ax^2 + 2bx + c = 0$ have a common root and the system of equations $ax + by = 0$, $cx + dy = 0$ have a non-trivial solution then prove that either a, b, c are in G.P. or b, c, d are in G.P.
10. Solve the following equations each of which has equal roots :
    (i) $4x^3 - 12x^2 - 15x - 4 = 0$,
    (ii) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$.
11. If $x^3 + 3x^2 - 9x + c$ is the product of three factors, two of which are identical, show that c is either 5 or -27 and resolve the given expression into factors in each case.

2.28 INTERMEDIATE VALUE THEOREM

If f(x) is a polynomial function and if $f(a) \neq f(b)$, then f(x) takes on every value between f(a) and f(b).

If f(a) and f(b) are of opposite signs then one root of the equation $f(x) = 0$ must lie between a and b.

As x changes gradually from a to b, the function f(x) changes gradually from f(a) to f(b), and therefore must pass through all intermediate values ; but since f(a) and f(b) have contrary signs the value zero must lie between them ; that is, f(x) = 0 for some value of x between a and b.

It does not follow that f(x) = 0 has only one root between a and b ; neither does it follow that if f(a) and f(b) have the same sign f(x) = 0 has no root between a and b.

Note

1. Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, provided the leading coefficient is positive.

In the function f(x) substitute for x the values $+\infty, 0, -\infty$ successively, then

$f(\infty) = \infty, f(0) = a_n, f(-\infty) = -\infty$;

If a_n is positive, then f(x) = 0 has a root lying between 0 and $-\infty$, and if p_n is negative f(x) = 0 has a root lying between 0 and ∞

STUDY TIP If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2}$ .... $+ a_n$, for sufficiently large values of x, f(x) has the same sign as a_0.

2. Every equation which is of an even degree and has its last term negative has at least two real roots, one positive and one negative, provided the leading coefficient is positive.

In the function f(x) substitute for x the values $+\infty, 0, -\infty$ successively, then

$f(\infty) = \infty, f(0) = a_n, f(-\infty) = \infty$;

Since a_n is negative, f(x) = 0 has a root lying between 0 and ∞ and a root lying between $-\infty$ and 0.

3. If the expressions f(a) and f(b) have contrary signs, an odd number of roots of f(x) = 0 will lie between a and b; and if f(a) and f(b) have the same sign, either no root or an even number of roots will lie between a and b

➤ **Example 1**. Show that the equation

$10x^3 - 17x^2 + x + 6 = 0$ has atleast one root between –1 and 0.

➤ ***Solution*** Let $f(x) = 10x^3 - 17x^2 + x + 6 = 0$

$f(-1) = -10 - 17 - 1 + 6 = -22 < 0$

$f(0) = 6 > 0$

Since f(–1). f(0) < 0, f(x) = 0 has atleast one root between –1 and 0.

STUDY TIP To determine the nature of roots of some equations the following statements are helpful:

(i) If the coefficients are all positive, the equation has no positive root ; thus the equation $x^5 + 4x^3 + 2x + 1 = 0$ cannot have a positive root.

(ii) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers of x are all of opposite sign, the equation has no negative root ; thus the equation
$x^7 + x^5 - 3x^4 + x^3 - 3x^2 + 2x - 5 = 0$
cannot have a negative roots.

(iii) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root ; thus the equation $x^8 + 3x^4 + 2x^2 + 1 = 0$ cannot have a real root.

(iv) If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except x = 0 ; thus the equation $x^9 + 4x^5 + 5x^3 + 3x = 0$ has no real root except x = 0.

2.29 DESCARTES' RULE OF SIGNS

Consider a polynomial equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2}$ .... $+ a_n$

with real coefficients and $a_n \neq 0$.

We have $f(-x) = a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2}$ .... $+ a_n$

By observing the sign variation in the coefficients of f(x) and f(–x) we can predict the following things about the nature of roots of f(x).

(i) The maximum number of positive roots of f(x) is equal to the number of sign changes in the coefficients of f(x). Let this number be 'p'.

(ii) The maximum number of negative roots of f(x) is equal to the number of sign changes in the coefficients of f(–x). Let this number be 'q'.

(iii) The minimum number of imaginary roots of f(x) is equal to n – (p + q), where n is the degree of f(x).

Note

1. In this topic, we consider equations of the type f(x) = 0 where f(x) is a polynomial, and 'equation' will mean an equation of this kind.
2. The position of a real root is its position on the real number line, and is determined roughly for a non-integral root by finding two consecutive integers between which the root lies.

➤ **Example 1.** Show that $x^7 + 5x^4 - 3x + k = 0$ has at least four imaginary roots.

➤ ***Solution***

Case I. k = 0.

The equation is $x^7 + 5x^4 - 3x = 0 \Rightarrow x(x^6 + 5x^3 - 3) = 0$.

Therefore, x = 0 is a root and the other roots are the roots of $x^6 + 5x^3 - 3 = 0$.

Let $f(x) = x^6 + 5x^3 - 3$. Therefore, $f(-x) = x^6 - 5x^3 - 3$.

Each of f(x) and f(–x) has only one variation in sign. Also f(0) =– 3. Therefore, f(x) = 0 has one positive and one negative root. Thus the given equation possesses only three real roots. Consequently the equation has four imaginary roots.

Case II. k > 0.

Let $f(x) = x^7 + 5x^4 - 3x + k$.

$\therefore \quad f(-x) = -x^7 + 5x^4 + 3x + k$.

The number of variations in f(x) and f(–x) is two and one respectively. So the number of real roots of f(x) = 0 will not exceed three. Hence the equation has at least four imaginary roots.

Case III. k < 0.

Let $f(x) = x^7 + 5x^4 - 3x + k$.

$\therefore \quad f(-x) = -x^7 + 5x^4 + 3x + k$.

The number of variations in f(x) and f(–x) is one and two respectively. So the number of real roots of f(x) = 0 will not exceed three.

Hence the equation has at least four imaginary roots.

➤ **Example 2.** Find the number of positive and negative roots of the equation $x^5 - x^4 + x^3 + 8x^2 + 2x - 2 = 0$.

➤ ***Solution*** Let $f(x) = x^5 - x^4 + x^3 + 8x^2 + 2x - 2$.

Then $f(-x) = -x^5 - x^4 - x^3 + 8x^2 - 2x - 2$.

f(x) has three variations and f(–x) has two variations. So the maximum number of positive and negative roots of f(x) = 0 may be three and two respectively.

Now $(x + 1) f(x) = x^6 + 9x^3 + 10x^2 - 2$.

Due to multiplication by (x + 1), the number of positive roots in f(x) = 0 will not be altered, but (x + 1)f(x) has one sign variation and changes sign in $(0, \infty)$. Therefore, f(x) = 0 has only one positive root.

Again f(0) < 0 , f(–1) > 0 and $f(-\infty) = -\infty$. These alternate signs definitely suggest that f(x) = 0 has two negative roots. Thus f(x) = 0 has one positive and two negative roots.

➤ **Example 3**. Find the number and position of the real roots of the equation

$$x^4 - 41x^2 + 40x + 126 = 0.$$

Substitute in f(x) the values 1, 2, 3, 4, 5, 6 in succession, and the signs will be +, +, –, –, –, +. Hence there is atleast one root between 2 and 3, and atleast one between 5 and 6; but by Descartes' Rule of Signs there cannot be more than two positive roots.

Hence there are two positive roots which lie between 2 and 3 and between 5 and 6 respectively.

We can find in a similar manner that there are two negative roots which lie between – 1 and –2 and between –6 and –7 respectively.

➤ **Example 4.** Find the number and position of the real roots of the equation

$$x^4 - 14x^2 + 16x + 9 = 0.$$

In this case we should easily find the two negative roots which lie between 0 and –1 and between – 4 and –5 respectively. The positive roots would, however, probably escape notice as they both lie between 2 and 3; it will in fact be found that f(2) > 0, $f\left(2\frac{1}{4}\right) < 0$, and f(3) > 0.

Practice Problems

1. Find the integral part of the greater root of the equation $x^3 + x^2 - 2x - 2 = 0$
2. Find the nature of the roots of the equation $3x^4 + 12x^2 + 5x - 4 = 0$
3. Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ has at least four imaginary roots.
4. What may be inferred respecting the roots of the equation $x^{10} - 4x^6 + x^4 - 2x - 3 = 0$?
5. Find the least possible number of imaginary roots of the equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$.
6. Use Descartes' rule of signs to show that :
   (i) If q is positive, $x^3 + qx + r = 0$ has only one real root.
   (ii) $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots.

2.30 ALGEBRAIC INTERPRETATION OF ROLLE'S THEOREM

Let f(x) be a polynomial having roots α and β where α < β so that we have f(α) = f(β) = 0. Also, a polynomial function is continuous and differentiable everywhere. Thus f(x) satisfies the conditions of Rolle's theorem. Consequently, there exists atleast one number γ ∈ (α, β) such that f '(γ) = 0 . In other words x = γ is a root of f '(x) = 0. Thus, Rolle's theorem can be interpreted algebraically as follows :

Between any two roots of a polynomial f(x), there is a always a root of its derivative f '(x).

➤ **Example 1.** If a, b, c ∈ R such that 2a + 3b + 6c = 0, show that the quadratic equation $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

➤ ***Solution*** Consider the polynomial

$$f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$$

We have f(0) = 0

and $$f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0$$

[∵ 2a + 3b + 6c = 0]

So, 0 and 1 are two roots of f(x) = 0. Therefore, f '(x) = 0 i.e. $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

Important Deductions

(1) If all the roots of f(x) = 0 are real, then all the roots of f′(x) = 0 are also real, and the roots of the latter equation separate those of the former. For if f(x) is of degree n, f′(x) is of degree n – 1, and a root of f′(x) = 0 exists in each of the n – 1 intervals between the n roots of f(x) = 0.

(2) If all the roots of f(x) = 0 are real, so also are those of f′(x) = 0, f″(x) = 0, f‴(x) = 0, ..., and the roots of any one of these equations separate those of the preceding equation.
This follows from (1).

(3) Not more than one root of f(x) = 0 can
   (i) lie between two consecutive roots of f′(x) = 0, or
   (ii) be less than the least of these, or
   (iii) be greater than the greatest of these.

For let $\beta_1, \beta_2, ..., \beta_r$ be the real roots of f′(x), any of which may be multiple roots, and suppose that $\beta_1 < \beta_2 < .... < \beta_r$. Let α_1, α_2 be real roots of f(x) = 0.
If $\alpha_1 = \alpha_2$, then α_1 is one of the set $\beta_1, \beta_2, ... \beta_r$
If $\alpha_1 \neq \alpha_2$, by Rolle's theorem, a root of f′(x) lies between α_1 and α_2.
Hence
   (i) if $\beta_1 < \alpha_1 < \alpha_2 < \beta_2$, then β_1 and β_2 cannot be consecutive roots,
   (ii) if $\alpha_1 < \alpha_2 < \beta_2$, then β_1 cannot be the least root of f′(x) = 0;
   (iii) if $\beta_r < \alpha_1 < \alpha_2$, then β_r cannot be the greatest.

Thus, not more than one root of f(x) = 0 can lie in any one of the open intervals

$(-\infty, \beta_1), (\beta_1, \beta_2), \ldots\ldots (\beta_{r-1}, \beta_r), (\beta_r, \infty)$.

(4) If $f'(x) = 0$ has r real roots, then $f(x) = 0$ cannot have more than $(r + 1)$ real roots.
If $f(x) = 0$ has no multiple root, none of the roots of $f'(x) = 0$ is a root of $f(x) = 0$, and the theorem follows from (3).
If $f(x) = 0$ has an m-multiple root, we regard this as the limiting case in which m roots tend to equality. Thus the theorem is true in all cases.

(5) If $f^{(r)}(x)$ is the r^{th} derivative of f(x) and the equation $f^{(r)}(x) = 0$ has some imaginary roots, then $f(x) = 0$ has atleast as many imaginary roots.
If follows from (4) that $f(x) = 0$ has at least as many imaginary roots as $f'(x) = 0$.

(6) If all the real roots $\beta_1, \beta_2, \ldots$ of $f'(x) = 0$ are known, we can find the number of real roots of $f(x) = 0$ by considering the signs of $f(\beta_1), f(\beta_2), \ldots\ldots$ A single root of $f(x) = 0$, or no root, lies between β_1 and β_2.

according as $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, or the same sign.

➤ **Example 2.** Find the character of the roots of
$f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$.

➤ ***Solution*** $f'(x) = 12x^3 - 24x^2 - 12x + 24 = 12(x^2 - 1)(x - 2)$.

The roots of $f'(x) = 0$ are $-1, 1, 2$.

When $x = -\infty$	-1	1	2	∞
$f(x) = \infty$	$-$	$+$	$+$	∞

Therefore $f(x) = 0$ has two real roots, one lying between $-\infty$ and -1 and the other between -1 and 1. The other two roots are imaginary.

➤ **Example 3.** If $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots, prove that k must lie between 8 and 11.

➤ ***Solution*** Let $f(x) = x^4 - 14x^2 + 24x - k$.

Then $f'(x) = 4x^3 - 28x + 24 = 4(x - 1)(x - 2)(x + 3)$.

$\therefore$ $f'(x) = 0$ has the roots $-3, 1, 2$.

By Rolle's theorem the position and sign of the roots of $f(x) = 0$ can be found out as follows.

$x = -\infty$	-3	1	2	∞
$f(x) = \infty$	$-117 - k$	$11 - k$	$8 - k$	∞

If $f(x) = 0$ has four real and unequal roots, we must have $-117 - k < 0$, $11 - k > 0$ and $8 - k < 0$, or, $k > -117$, $k < 11$, $k > 8$. Therefore $8 < k < 11$.

➤ **Example 4.** Show that the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 + (x - d)^3 = 0$, where a, b, c, d are not all equal, has only one real root.

➤ ***Solution*** Let $f(x) = (x - a)^3 + (x - b)^3 + (x - c)^3 + (x - d)^3$.

Then $f'(x) = 3[(x - a)^2 + (x - b)^2 + (x - c)^2 + (x - d)^2]$.

$f'(x)$ remains positive for any real value of x. Therefore, $f'(x) = 0$ has only one real root.

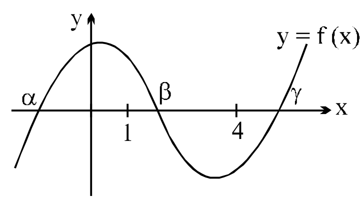
➤ **Example 5**. Given the cubic equation $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one root of the equation is less than 1, other root is in the interval (1, 4) and the third root is greater than 4, then the value of k lies in the interval $\left(a + \sqrt{b}, b(a + \sqrt{b})\right)$ where $a, b \in N$. Find the value of a and b.

➤ ***Solution*** $f(x) = x^3 - 2kx^2 - 4kx + k^2 = 0$

note that $f(0) = k^2 > 0$

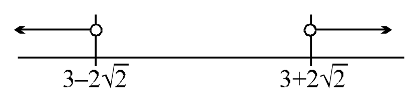
$f(1) > 0$

$\Rightarrow$ $1 - 2k - 4k + k^2 > 0$



$k^2 - 6k + 1 > 0$

$[k - (3 + 2\sqrt{2})][k - (3 - 2\sqrt{2})] > 0$



....(1)

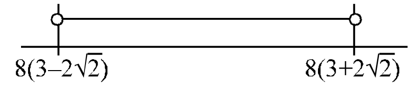
Also $f(4) < 0$

$\Rightarrow$ $64 - 32k - 16k + k^2 < 0$ $\quad k^2 - 48k + 64 < 0$

$(k - 24)^2 < 512$

$(k - 24 + 16\sqrt{2})(k - 24 - 16\sqrt{2}) < 0$

$[k - 8(3 - 2\sqrt{2})][k - 8(3 + 2\sqrt{2})] < 0$



....(2)

$(1) \cap (2) \Rightarrow 3 + 2\sqrt{2} < k < 8(3 + 2\sqrt{2})$

$3 + \sqrt{8} < k < 8(3 + \sqrt{8})$

$\therefore$ $a = 3;\ b = 8$

Practice Problems

1. If a, b, c $\in$ R and a + b + c = 0 then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has atleast one root in (0, 1).
2. If f(x) = (x – 1)(x – 2)(x – 3)(x – 4) , find the number of real roots of f′(x) = 0 and indicate the intervals in which they lie.
3. If $f(x) = x^2(1-x)^3$ then prove that the equation f′(x) = 0 has atleast one root in (0, 1).
4. Prove that the equation $x^4 - 4x - 2 = 0$ cannot have more than two real roots.
5. Prove that the equation $x^5 + x^3 + 4x + 1 = 0$ has atleast four imaginary roots.

Target Problems for JEE Advanced

➤ **Problem 1.** Find a quadratic equation whose roots x_1 and x_2 satisfy the condition

$$x_1^2 + x_2^2 = 5, 3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3).$$

(Assume that x_1, x_2 are real)

➤ ***Solution*** We have $3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$

$$\Rightarrow \frac{x_1^5 + x_2^5}{x_1^3 + x_2^3} = \frac{11}{3}$$

$$\Rightarrow \frac{(x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2x_2^2(x_1 + x_2)}{(x_1^3 + x_2^3)} = \frac{11}{3}$$

$$\Rightarrow (x_1^2 + x_2^2) - \frac{x_1^2x_2^2(x_1 + x_2)}{(x_1 + x_2)(x_1^2 + x_2^2 - x_1x_2)} = \frac{11}{3}$$

$$\Rightarrow 5 - \frac{x_1^2x_2^2}{5 - x_1x_2} = \frac{11}{3}$$

$$\Rightarrow 3x_1^2x_2^2 + 4x_1x_2 - 20 = 0$$

$$\Rightarrow 3x_1^2x_2^2 + 10x_1x_2 - 6x_1x_2 - 20 = 0$$

$$\Rightarrow (x_1x_2 - 2)(3x_1x_2 + 10) = 0$$

$\therefore$ $x_1x_2 = 2, -\frac{10}{3}$

We have $(x_1 + x_2)^2 = 5 + 4$

(if $(x_1x_2 = 2)$) $= 9$

$\therefore$ $x_1 + x_2 = \pm 3$

$\therefore$ $(x_1 + x_2)^2 = 5 + 2(-10/3)$

(if $x_1x_2 = -10/3$) $= 5/3$

which is not possible x_1, x_2 are real

Thus required quadratic equations are

$x^2 \pm 3x + 2 = 0$

➤ **Problem 2.** If the roots of the equation $ax^2 + 2bx + c = 0$ are real and distinct, then show that the roots of the equation $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$ are imaginary numbers and vice-versa.

➤ ***Solution*** Given equations are

$ax^2 + 2bx + c = 0$ ...(1)

and $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$

or $(a^2 - ac + 2b^2)x^2 + 2(a + c)bx + c^2 - ac + 2b^2 = 0$ ...(2)

Let D_1 and D_2 be the discriminants of equations (1) and (2) respectively, then

$D_1 = 4(b^2 - ac) = 4k$, where $k = b^2 - ac$

According to question $D_1 > 0$ ...(3)

Now equation (2) becomes,

$(a^2 + b^2 + k)x^2 + 2(a + c)bx + b^2 + c^2 + k = 0$ ...(4)

Now $D_2 = 4(a + c)^2b^2 - 4(a^2 + b^2 + k)(b^2 + c^2 + k)$

$= 4[a^2b^2 + c^2b^2 + 2acb^2 - (a^2 + b^2)(b^2 + c^2) - k(a^2 + b^2 + b^2 + c^2) - k^2]$

$= 4[a^2b^2 + c^2b^2 + 2acb^2 - a^2b^2 - b^4 - a^2c^2 - b^2c^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[2acb^2 - b^4 - a^2c^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[-(b^4 + a^2c^2 - 2acb^2) - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[-k^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$[\because b^4 + a^2c^2 - 2acb^2 = (b^2 - ac)^2 = k^2]$

$= -4k[2k + a^2 + c^2 + 2b^2] = -D_1[2(b^2 - ac) + a^2 + c^2 + 2b^2]$

$= -D_1[4b^2 + (a - c)^2]$

$\therefore$ $D_2 = -D_1(\text{a positive number}) < 0$ $[\because D_1 > 0]$

$4b^2 + (a - c)^2 = 0 \Rightarrow b = 0$ and $a = c$

$\Rightarrow$ $D_1 = 4(0^2 - a.a) = -4a^2 < 0$ not possible from (3)

➤ **Problem 3.** The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\dfrac{m}{n}, \infty\right)$ where *m* and *n* are relatively prime positive integers, find the value of $(m+n)$.

➤ ***Solution*** We have $a^2-(2x^2+1)a+x^4+x=0$

$$\therefore \quad a=\frac{(2x^2+1)\pm\sqrt{(2x^2+1)^2-4(x^4+x)}}{2}$$

$$\therefore \quad 2a=(2x^2+1)\pm\sqrt{4x^2-4x+1}$$
$$=(2x^2+1)\pm(2x-1)$$

+ ve sign $\quad a=x^2+x \quad$ – ve sign

$2a=2x^2-2x+2 \quad a=x^2-x+1$

if $x^2+x-a=0 \Rightarrow x=\dfrac{-1\pm\sqrt{1+4a}}{2}$

if $x^2-x+1-a \Rightarrow x=\dfrac{1\pm\sqrt{1-4+4a}}{2}=\dfrac{1\pm\sqrt{4a-3}}{2}$

for x to be real $a \geq 3/4$ and $a \geq -1/4$

$\Rightarrow \quad a \geq 3/4$

$\Rightarrow \quad 3+4=7$

➤ **Problem 4.** Find all positive integers a, b such that each of the equations $x^2-ax+b=0$ and $x^2-bx+a=0$ has distinct positive integral roots.

➤ ***Solution*** Let integers $\alpha>\beta>0$ be the roots of

(i) $\quad x^2-ax+b=0$ and let integers $\gamma>\delta>0$ be the roots of

(ii) $\quad x^2-bx+a=0$. For definiteness, let $a \geq b$. Now $\alpha+\beta=a$, $\alpha\beta=b$, and $\gamma+\delta=b$, $\gamma\delta=a$.

Hence $\alpha-\beta=1-(\alpha-1)(\beta-1)$. Hence $0 \leq 1-(a-1)(b-1) \leq 1$. So $\beta=1$ since α, β are positive integers and $\beta \leq \alpha$. Thus $a-b=1$. Further, $a-b=(\gamma-1)(\delta-1)-1$, so that $(\gamma-1)(\delta-1)=2$. So since $\gamma>\delta>0$ are integers, we see that $\gamma-1=2$ and $\delta-1=1$, so that $\gamma=3$, $\delta=2$.

Hence, $a=\gamma\delta=6$ and $b=\gamma+\delta=5$. Also, therefore, $\alpha=5$, $\beta=1$.

➤ **Problem 5.** Find all positive integers n for which n^2+96 is a perfect square.

➤ ***Solution*** Suppose m is a positive integer, such that

$$n^2+96=m^2.$$

Then $\quad m^2-n^2=96$,

i.e., $\quad (m-n)(m+n)=96$.

Since $m-n<m+n$, and $m-n$, $m+n$ must be both even (because $m+n=(m-n)+2n$ therefore $m-n$, $m+n$ must be both odd or both even; also if both of them are odd, then the product cannot be even).

Therefore the only possibilities are

$m-n=2, m+n=48$; $\quad m-n=4, m+n=24$;
$m-n=6, m+n=16$; $\quad m-n=8, m+n=12$.

The corresponding solutions are

$m=25, n=23$; $\quad m=14, n=10$;
$m=11, n=5$; $\quad m=10, n=2$.

➤ **Problem 6.** Find all integers values of a such that the quadratic expressions $(x+a)(x+1991)+1$ can be factored as $(x+b)(x+c)$, where b and c are integers.

➤ ***Solution*** $(x+a)(x+1991)+1=(x+b)(x+c)$

$\Rightarrow \quad 1991+a=b+c$

and $\quad 1991a+1=bc$

$\therefore \quad (b-c)^2=(b+c)^2-4bc=(1991+a)^2-(1991a+1)$

$$=\underbrace{(1991+a)^2-4\times1991a}-4$$

$$=(1991-a)^2-4$$

or $\quad (1991-a)^2=(b-c)^2=4$

If the difference between two perfect square is 4, then one of them is 4 and the other is zero.

(Prove this)

Therefore, $1991-a=\pm2$, $\quad (b-c)^2=0$

$\Rightarrow \quad a=1991+2=1993 \quad$ and $b=c$

or $\quad a=1991-2=1989 \quad$ and $b=c$

But $b+c=2b=1991+a$

$=1991+1993$ or $1991+1989$

$\Rightarrow \quad b=c=1992$ or 1990

So, the only 2 values of a are 1993 and 1989.

➤ **Problem 7.** Let a & c be prime numbers and b an integer. Given that the quadratic equation $ax^2+bx+c=0$ has rational roots, show that one of the root is independent of the coefficients. Find the two roots .

➤ ***Solution*** $b^2-4ac=n^2 \Rightarrow (b-n)(b+n)=4ac$

Case I $\quad b-n=4a$ and $b+n=c$
Case II $\quad b-n=4c$ and $b+n=a$
not possible as $2b=4a+c=$ odd

$\Rightarrow \quad$ b is not an integer

Case III $\quad b-n=2a \quad$ and $\quad b+n=2c$

$\Rightarrow \quad b = a + c$

Now $\quad \alpha\beta = c/a \quad \& \quad \alpha + \beta = -b/a$

$$= -\frac{a + c}{a} = -1 - c/a$$

$\Rightarrow \quad \alpha = -1 \quad$ and $\quad \beta = -c/a$

➤ **Problem 8.** If $x_1 \in N$ and x_1 satisfies the equation. If $x^2 + ax + b + 1 = 0$, where a, $b \neq -1$ are integers has a root in natural numbers then prove that $a^2 + b^2$ is a composite.

➤ ***Solution*** Let α and β be the two roots of the equation where $\alpha \in N$. Then

$\alpha + \beta = -a \qquad$ ...(1)

$\alpha . \beta = b + 1 \qquad$ ...(2)

$\therefore \quad \beta = -a - \alpha$ is an integer. Also, since $b + 1 \neq 0$, $\beta \neq 0$.

From Eq. (1) and Eq. (2), we get

$a^2 + b^2 = (\alpha + \beta)^2 + (\alpha\beta - 1)^2 = \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1$

$= (1 + \alpha^2)(1 + \beta^2)$

Now, as $\alpha \in N$ and β is a non–zero integer, $1 + \alpha^2 > 1$ and $1 + \beta^2 > 1$.

Hence $a^2 + b^2$ is composite number

Note

If $b = -1$, then $a^2 + b^2$ can not be a composite number.

Consider $a = -6$, $b = -1$

$x^2 - 6x + (-1) + 1 = 0$, its are 6 and 0.

$a^2 + b^2 = 36 + 1 = 37$, a prime number.

➤ **Problem 9.** If a, b, c are odd integers, show that the roots of the equation $ax^2 + bx + c = 0$ cannot be rational.

➤ ***Solution*** Let $a = 2m + 1$, $b = 2n + 1$, $c = 2p + 1$

Now $D = (2n + 1)^2 - 4(2m + 1)(2p + 1) \qquad$ ...(1)

= (an odd integer) – (an even integer) = an odd integer

If possible let D be a perfect square of a rational number. Since D is an odd integer and is the square of a rational number therefore D must be the square of an odd integer.

Let $D = (2k + 1)^2$

Now from (1), $(2k + 1)^2 = (2n + 1)^2 - 4(2m + 1)(2p + 1)$

or $\quad 4(2m + 1)(2p + 1) = (2n + 1)^2 - (2k + 1)^2$

$= (2n + 1 + 2k + 1)(2n + 1 - 2k - 1)$

or $\quad (2m + 1)(2p + 1) = (n + k + 1)(n - k) \qquad$ ...(2)

If n and k are both odd or both even integers, then $n - k$ will be an even integer and if one of n and k is an odd integer and other an even integer, then $n + k + 1$ will be an even integer. Thus R. H. S. of equation (2) is an even integer and L.H.S. is an odd integer which is not possible. Hence D cannot be a perfect square i.e. the roots of equation $ax^2 + bx + c = 0$ cannot be rational.

➤ **Problem 10.** Show that the quadratic equation $x^2 + 7x - 14(m^2 + 1) = 0$, where m is an integer, has no integral roots.

➤ ***Solution*** Given equation is $x^2 + 7x - 14(m^2 + 1) = 0 \qquad$ ...(1)

Let α and β be the roots of equation (1).

then $\alpha + \beta = -7 \qquad$ ...(2)

If possible, let α and β be integers, $\alpha\beta = 14(m^2 + 1) \qquad$ ...(3)

From (3), $\dfrac{\alpha . \beta}{7} = 2(m^2 + 1)$ = an integer

$\therefore \quad \beta\alpha$ is divisible by 7 and 7 is a prime number

$\therefore \quad$ at least one of α and β must be a multiple of 7.

Let $\alpha = 7k$, then from (2),

$= -7 - 7k = -7(k + 1)$ = a multiple of 7.

From (3), $7k\{-7(k + 1)\} = 14(m^2 + 1)$

or $\quad -\dfrac{2(m^2 + 1)}{7} = k(k + 1)$ = an integer $\qquad$ ...(4)

If m = 0, L.H.S. of (4) is not an integer. Since m^2 occurs, therefore we may assume m to be a positive integer.

Now we claim that $m^2 + 1$ is not divisible by 7 for all $m \in N$.

Let $f(m) = m^2 + 1$

Let P(m) : f(m) is not divisible by 7

Now $f(1) = 1^2 + 1 = 2$, which is not divisible by 7.

Similarly, we can show that f(2), f(3), f(4), f(5), f(6), f(7) are not divisible by 7.

Hence P(1), P(2),......., P(7) are true $\qquad$ ...(A)

Let P(m) be true $\Rightarrow$ f(m) i.e. $m^2 + 1$ is not divisible by 7. $\qquad$ ...(5)

Now $f(m + 7) = (m + 7)^2 + 1 = m^2 + 1 + (14m + 49)$, which is not divisible by 7 as $(m^2 + 1)$ is not divisible by 7

Hence P(m + 7) is true whenever P(m) is true ...(B)
From (A) and (B) it follows that $m^2 + 1$ is not divisible by 7.
Thus from (4), we get a contradiction.

Hence α, β cannot be integers.

Since, $\alpha + \beta = -7$ = an integer, therefore exactly one of α and β cannot be non-integer.

➤ **Problem 11.** Let a, b and c be integers with a > 1, and let p be a prime number. Show that if $ax^2 + bx + c$ is equal to p for two distinct integral values of x, then it cannot be equal to 2p for any integral value of x.

➤ ***Solution*** Given, $ax^2 + bx + c - p = 0$ for two distinct integral values of x say α and β. Then α, β are the roots of equation $ax^2 + bx + c - p = 0$ ...(1)

$\therefore \quad \alpha + \beta = -\dfrac{b}{a}$ ...(2)

and $\alpha\beta = \dfrac{c-p}{a}$ ...(3)

To prove $ax^2 + bx + c - 2p \neq 0$ for any integral value of x.
If possible, let $ax^2 + bx + c - 2p = 0$ for some integer k, then

$ak^2 + bk + c - 2p = 0$ or $ak^2 + bx + c - p = p$

or $k^2 + \dfrac{b}{a}k + \dfrac{c-p}{a} = \dfrac{p}{a}$ or $k^2 - (\alpha + \beta)k + \alpha\beta = \dfrac{p}{a}$

or $(k - \alpha)(k - \beta) = \dfrac{p}{a}$ or $\dfrac{p}{a} = (k - \alpha)(k - \beta)$

= an integer ...(4)

Since p is a prime number and from (4), $\dfrac{p}{a}$ is an integer

$\therefore \quad a = p$ or $a = 1 \quad \Rightarrow \quad a = p \quad [\because a > 1]$
Now from (4), $(k - \alpha)(k - \beta) = 1 \quad [\because p = a]$
$\therefore$ either $k - \alpha = 1$ and $k - \beta = \Rightarrow \alpha = \beta$ (not acceptable as $\alpha \neq \beta$)
or $k - \alpha = -1$ and $k - \beta = -1 \Rightarrow \alpha = \beta$ (not possible)
Hence $p \neq a$.

Now from (4), $(k - \alpha)(k - \beta) = \dfrac{p}{a}$ (not possible)

Thus we get a contradiction.

Hence $ax^2 + bx + c \neq 2p$ for any integral value of x.

➤ **Problem 12.** If each pair of the following three equations

$x^2 + ax + b = 0,\ x^2 + cx + d = 0,\ x^2 + ex + f = 0$

has exactly one root in common, then show that

$(a + c + e)^2 = 4\ (ac + ce + ea - b - d - f)$

➤ ***Solution*** $x^2 + ax + b = 0$ ...(1)

$x^2 + cx + d = 0$ ...(2)

$x^2 + ex + f = 0$ ...(3)

Let α, β be the roots of (1), β, γ be the roots of (2) and γ, δ be the roots of (3).

$\therefore \quad \alpha + \beta = -a,\ \alpha\beta = b$ ...(4)

$\beta + \gamma = -c,\ \beta\gamma = d$ ...(5)

$\gamma + \alpha = -e,\ \gamma\alpha = f$ ...(6)

$\therefore$ L.H.S. $= (a + c + e)^2 = (-\alpha - \beta - \gamma - \gamma - \alpha)^2$ {from (4), (5) & (6)}

$= 4(\alpha + \beta + \gamma)^2$ ...(7)

R.H.S. $= 4(ac + ce + ea - b - d - f)$

$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha)\}$ {from (4), (5) & (6)}

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$

$= (\alpha + \beta + \gamma)^2$ ...(8)

From (7) & (8),

$(a + c + e)^2 = 4\ (ac + ce + ea - b - d - f)$

➤ **Problem 13.** If the three equation $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common possible root then find a and b and the roots.

➤ ***Solution*** Let α be the common root of the three equations are their other roots be β, γ, δ respectively

$\therefore \quad \alpha + \beta = -a,\ \alpha\beta = 12$

$\alpha + \gamma = -b,\ \alpha\gamma = 15$

$\alpha + \delta = -(a + b),\ \alpha\delta = 36$

$\therefore \quad (\alpha + \beta) + (\alpha + \gamma) = -(a + b) = \alpha + \delta$

$\therefore \quad \alpha + \beta + \gamma = \delta$ ...(i)

Again $\alpha(\beta + \gamma + \delta) = 12 + 15 + 36 = 63$

or $\quad \alpha(2\delta - \alpha) = 63$ by (i)

or $\quad 2\alpha\delta - \alpha^2 - 63$ or $72 - \alpha^2 = 63$

$\because \quad \alpha\delta = 36$

$\therefore \quad \alpha^2 = 9 \quad \therefore \quad \alpha = 3, -3$

$\alpha = 3 \quad \Rightarrow \quad \beta = 4, \gamma = 5, \delta = 12$

$\alpha = -3 \quad \Rightarrow \quad \beta = -4, \gamma = -5, \delta = -12$

$\therefore \quad a = -(\alpha + \beta) = 7, -7 \; ; \; b = -(\alpha + \gamma) = 8, -8$

➤ **Problem 14.** If α is a root of the equation $ax^2 + bx + c = 0$, β is a root of the equation $-ax^2 + bx + c = 0$ and γ is a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ then prove that γ lies between α and β.

➤ ***Solution*** For a = 0, the three given equations become linear in x and has a single root so that $\alpha = \beta = \gamma$

Since $\quad a\alpha^2 + b\alpha + c = 0 \qquad$ ...(1)

we have $\frac{a}{2}\alpha^2 + b\alpha + c = (a\alpha^2 + b\alpha + c) - \frac{a\alpha^2}{2}$

$= -\frac{a\alpha^2}{2} \qquad$ {from (1)}

and also $\quad -a\beta^2 + b\beta + c = 0 \qquad$ ...(2)

then $\frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3}{2}a\beta^2$

$= \frac{3}{2}a\beta^2 \qquad$ {from (2)}

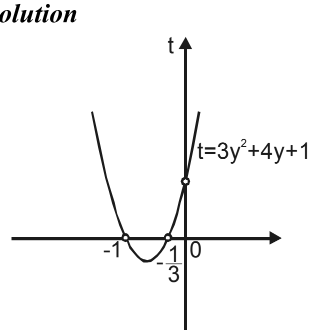
Consequently, the expressions $\frac{a}{2}\alpha^2 + b\alpha + c = -\frac{a\alpha^2}{2}$ and $\frac{a}{2}\beta^2 + b\beta + c = \frac{3}{2}a\beta^2$ are of different signs. Hence the graph of $f(x) = \frac{a}{2}x^2 + bx + c$ intersects the x-axis at $(\gamma, 0)$. Hence γ is a root where

$\alpha < \gamma < \beta$ or $\alpha > \gamma > \beta$.

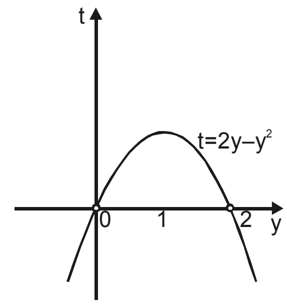
➤ **Problem 15.** Find the integral solutions of the equation $x^2 + x = y^4 + y^3 + y^2 + y$

➤ ***Solution***

(a)



(b)



Let us multiply both members of the equation by 4 and add 1 to them ; this results in the equivalent equation

$(2x + 1)^2 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$

whose left-hand member is a perfect square. Further, we have

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (4y^4 + 4y^3 + y^2)$
$+ (3y^2 + 4y + 1) = (2y^2 + y)^2 + (3y^2 + 4y + 1)$
$= (P(y))^2 + Q(y)$

Since the quadratic trinomial $Q(y) = 3y^2 + 4y + 1$ possesses (real) roots $y_1 = -1$ and $y_2 = -1/3$ it assumes positive values for all integral values of y different from $y = -1$ (see the graph of the function $t = 3y^2 + 4y + 1$ in figure (a). Therefore $(2x + 1)^2 > (P(y))^2 = (2y^2 + y)^2$.

On the other hand,

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y + 1)^2 + (2y - y^2)$
$= (P_1(y))^2 + Q_1(y).$

The graph of the function $Q_1(y) = 2y - y^2$ is shown in figure (b) ; the roots of the quadratic binomial $Q_1(y)$ are equal to 0 and 2.

Therefore $Q_1(y) < 0$ for all integral values of y different from 0, 1 and 2, whence $(2x + 1)^2 < (P_1(y))^2 = (2y^2 + y + 1)^2$.

Thus, for all integral values of y different from –1, 0, 1 and 2 there hold the inequalities

$(2y^2 + y + 1)^2 > (2x + 1)^2 > (2y^2 + y)^2$

This means that for such y the number $(2x + 1)^2$ lies between the squares of the two consecutive whole numbers Q(y) and $Q_1(y)$, and therefore 2x + 1 cannot be equal to an integral number.

Thus, in case y is an integral number, the number x can be integral only when y is equal to –1, 0, 1 or 2, that is when the right-hand side of the original equation is equal to 0, 0, 4 or 30 respectively. It now remains to solve 3 quadratic equations of the form

$x^2 + x = c$ where c is equal to 0, 4 or 30 (1)

These equations have the following integral roots :

$x = 0$ and $x = -1$ for $c = 0$; $x = 5$ and $x = -6$ for $c = 30$; for $c = 4$ equation (1) has no integral roots

Hence, finally, we arrive at the following set of integral solutions of the given equation :

(0, –1), (–1, –1) ; (0, 0), (–1, 0) ; (5, 2), (–6, 2)

➤ **Problem 16.** A quadratic trinomial $p(x) = ax^2 + bx + c$ is such that $|p(x)| \le 1$ for $|x| \le 1$. Prove that in this case from the condition $|x| \le 1$ it also follows that $|p_1(x)| \le 2$ where $p_1(x) = cx^2 + bx + a$.

➤ ***Solution*** Let us assume that $a \ge 0$ (if otherwise, we can replace the polynomial p(x) by the polynomial $-p(x) = -ax^2 - bx - c$ satisfying the same conditions). We shall also assume that $b \ge 0$ (if otherwise, we can replace p(x) by $p(-x) = ax^2 - bx + c$). Now we substitute the values $x = 1$, $x = 0$ and $x = -1$ into the inequality $|p(x)| = ax^2 + bx + c| \le 1$, which results in

$|a + b + c|\ 1 \le$, $|c| \le 1$ and $|a - b + c| \le 1$,

that is $|c| \le 1$ and $|a + b| \le 2$, $|a - b| \le 2$

Further, if $c \ge 0$ then $0 \le cx^2 \le c$ for $|x| \le 1$; for $|x| \le 1$ we also have $-b \le bx \le b$. This means that for these values of x there hold the relations

$p_1(x) = cx^2 + bx + a \le c + b + a \le 1$

and $p_1(x) = cx^2 + bx + a \ge 0 + (-b) + a = a - b \ge -2$

whence it follows that $|p_1(x)| \le 2$. Similarly, if $c \le 0$ and $c \le cx^2 \le 0$ (and, as before, $-b \le bx \le b$) then

$p_1(x) = cx^2 + bx + a \le 0 + b + a = a + b \le 2$

and $p_1(x) = cx^2 + bx + a \ge c + (-b) + a = a - b + c \ge -1$

whence it follows that $|p_1(x)| \le 2$ in this case as well.

➤ **Problem 17.** Find out the values of 'a' for which any solution of the inequality, $\dfrac{\log_3 (x^2 - 3x + 7)}{\log_3 (3x + 2)} < 1$ is also a solution of the inequality, $x^2 + (5 - 2a)x \le 10a$.

➤ ***Solution*** Note that $x^2 - 3x + 7 > 0 \ \forall \ x \in R$

Also $x > -2/3$ and $x \ne -1/3$

Also $3x + 2 > 1 \Rightarrow \ x > -1/3$

and $0 < 3x + 2 < 1 \quad \Rightarrow \ -2/3 < x < -1/3$

Hence for $x > -1/3$

$\log_{3x+2} (x^2 - 3x + 7) < 1$

$\Rightarrow \ x^2 - 3x + 7 < 3x + 2$

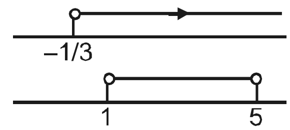
$x^2 - 6x + 5 < 0 \qquad (x - 5)(x - 1) < 0$

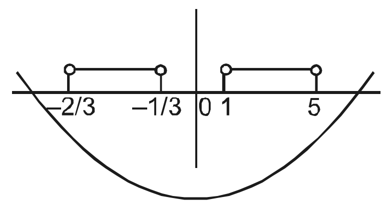
$\Rightarrow \ x \in (1, 5)$

for $-2/3 < x < -1/3$

$$x \in \left(-\frac{2}{3}, -\frac{1}{3}\right)$$

Hence the solution of the first inequality is





$$x \in \left(-\frac{2}{3}, -\frac{1}{3}\right) \cup (1, 5)$$

Now if any solution of the inequality is also the solution of,

$f(x) = x^2 + (5 - 2a)x - 10a \le 0$ then

$f(5) \le 0$ and

$$f\left(-\frac{2}{3}\right) \le 0 \qquad \Rightarrow \quad a \ge 5/2$$

➤ **Problem 18.** Find the set of values of 'y' for which the inequality ,

$2\log_{0.5} y^2 - 3 + 2x \log_{0.5} y^2 - x^2 > 0$

is valid for atleast one real value of 'x' .

➤ ***Solution*** $-4\log_2 |y| - 3 - 4\log_2 |y|\, x - x^2 > 0$

or $x^2 + 4\log_2 |y|\, x + 3 + 4\log_2 |y| < 0$

or $x^2 + 4tx + 3 + 4t < 0$

for this inequality to be true for atleast one x, $3 > 0$

$\Rightarrow \ 4t^2 - 4t - 3 > 0 \qquad (2t - 3)(2t + 1) > 0$

$$\Rightarrow \ t > \frac{3}{2} \quad \text{or} \quad t < -\frac{1}{2} \quad \text{i.e.} \quad \log_2 |y| > \frac{1}{2}$$

$\Rightarrow \ |y| > 2\sqrt{2}$

$\Rightarrow \ y > 2\sqrt{2}$ or $y < 2\sqrt{2}$

$$\text{Similarly} \quad \log_2 |y| < -\frac{1}{2} \qquad |y| < \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad y < \dfrac{1}{\sqrt{2}} \quad$ or $\quad y > -\dfrac{1}{\sqrt{2}}$

$$y \in \left(-\infty, -2\sqrt{2}\right) \cup \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(2\sqrt{2}, \infty\right)$$

➤ **Problem 19.** Find the values of 'p' for which the inequality, $\left(2 - \log_2\left(\frac{p}{p+1}\right)\right) x^2 + 2x \left(1 + \log_2 \frac{p}{p+1}\right) - 2\left(1 + \log_2 \frac{p}{p+1}\right) > 0$ is valid for all real x.

➤ ***Solution*** $(2 - t)x^2 + 2(1 + t)x - 2(1 + t) > 0$

when $t = 2$, $6x - 6 > 0$ which is not true $\forall$ $x \in R$.

Let $t \neq 2$; $t < 2$ ...(1)

and $4(1 + t)^2 + 8(1 + t)(2 - t) < 0$

(for given inequality to be valid)

or $(t - 5)(t + 1) > 0$

$\Rightarrow$ $t > 5$ or $t < -1$ ...(2)

From (1) and (2) ; $t < -1$

$\Rightarrow \quad \log_2 \dfrac{p}{p+1} < -1$

$\dfrac{p}{p+1} < \dfrac{1}{2}$ or $\dfrac{p-1}{p+1} < 0$

$\Rightarrow \quad -1 < p < 1$

but $\dfrac{p}{p+1} > 0 \quad \Rightarrow \quad p > 0$ or $p < -1$.

common solution is $p \in (0, 1)$

➤ **Problem 20.** If $ax^2 - bx + c = 0$ have two distinct roots lying in the interval (0, 1), a, b, c $\in$ N, then prove that $\log_5 abc \geq 2$.

➤ ***Solution*** Let roots are α and β

$\therefore \quad \alpha + \beta = b/a$ and $\alpha\beta = c/a$

and $0 < \alpha < 1, 0 < \beta < 1$

$\therefore \quad 0 < 1 - \alpha < 1$ & $< 1 - \beta < 1$

then $\dfrac{\alpha + (1-\alpha)}{2} \geq \sqrt{\alpha(1-\alpha)}$ {$\because$ A.M. $\geq$ G.M.}

$\Rightarrow \quad \dfrac{1}{4} \geq \alpha(1 - \alpha) > 0$

similarly $\dfrac{1}{4} \geq \beta(1 - \beta) > 0$

$\therefore \quad \dfrac{1}{16} \geq \alpha\beta(1 - \beta) > 0$

But α and β are distinct.

$\therefore \quad 0 < \alpha\beta(-\alpha)(1 - \beta) < \dfrac{1}{16}$

$\Rightarrow \quad 0 < \alpha\beta(1 - (\alpha + \beta) + \alpha\beta) < \dfrac{1}{16}$

$\Rightarrow \quad 0 < \dfrac{c}{a}\left(1 - \dfrac{b}{a} + \dfrac{c}{a}\right) < \dfrac{1}{16}$

$\Rightarrow \quad 0 < c(a - b + c) < \dfrac{a^2}{16}$

$\because \quad c(a - b + c) =$ Natural Number ($\because$ a, b, c $\in$ N)

$\therefore \quad$ Minimum value of $c(a - b + c) = 1$

$\therefore \quad \dfrac{a^2}{16} > 1, \quad \therefore a \geq 5 \qquad (\because a \in N)$

and condition for real roots, $b^2 - 4ac \geq 0$

$\Rightarrow \quad b^2 \geq 4ac$

$\Rightarrow \quad b^2 \geq 2ac \qquad (\because a \geq 5)$

$\Rightarrow \quad b \geq 5 \qquad (\because b \in N)$

and minimum value of c = 1

Hence $\quad abc \geq 25$

$\log_5(abc) \geq \log_5 5^2$

$\log_5(abc) \geq 2$

➤ **Problem 21.** Show that for all real values of x, then expression $\dfrac{2a(x-1)\sin^2\alpha}{x^2 - \sin^2\alpha}$ cannot lie between $2a\sin^2\dfrac{\alpha}{2}$ and $2a\cos^2\dfrac{\alpha}{2}$.

➤ ***Solution*** Let $y = \dfrac{2a(x-1)\sin^2\alpha}{x^2 - \sin^2\alpha}$

or $yx^2 - 2a\sin^2\alpha x + + (2a - y)\sin^2\alpha = 0$ ...(1)

Since x is real $\therefore$ $D \geq 0$

$\therefore$ $4a^2\sin^4\alpha - 4y(2a - y)\sin^2\alpha \geq 0$

or $a^2\sin^2\alpha - y(2a - y) \geq 0$ [Dividing by $4\sin^2\alpha$]

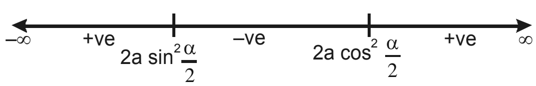
or $y^2 - 2ay + a^2\sin^2\alpha \geq 0$ ...(2)

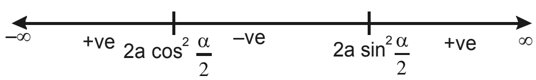
Now $y^2 - 2ay + a^2\sin^2\alpha = 0$

$$\Rightarrow \quad y = \frac{2a \pm \sqrt{4a^2 - 4a^2\sin^2\alpha}}{2}$$

$$\Rightarrow \quad y = a \pm a\cos\alpha = a(1 \pm \cos\alpha) = 2a\sin^2\frac{\alpha}{2}, 2a\cos^2\frac{\alpha}{2}$$

sign scheme for $y^2 - 2ay + a^2\sin^2\alpha$ is as follows





$\therefore$ $y^2 - 2ay + a^2\sin^2\alpha \geq 0 \Leftrightarrow$ y does not lie between

$2a\sin^2\dfrac{\alpha}{2}$ and $2a\cos^2\dfrac{\alpha}{2}$

or, $2a\cos^2\dfrac{\alpha}{2}$ and $2a\sin^2\dfrac{\alpha}{2}$ as the case may be.

➤ **Problem 22.** Prove that if x is real, the minimum value of $\dfrac{(a+x)(b+x)}{(c+x)}$ $(x > -c)$, for $x > c, b > c$ is $(\sqrt{(a-c)} + \sqrt{(b-c)})^2$.

➤ ***Solution*** Let $y = \dfrac{(a+x)(b+x)}{(c+x)}$

$\Rightarrow$ $x^2 + (a + b)x + ab = cy + xy$

$\Rightarrow$ $x^2 + (a + b + y)x + ab - cy = 0$

for real x, $B^2 - 4AC \geq 0$

$\Rightarrow$ $(a + b + y)^2 - 4ab + 4cy \geq 0$

$\Rightarrow$ $(a + b)^2 y^2 - 2(a + b)y - 4ab + 4cy \geq 0$

$\Rightarrow$ $(a - b)^2 + y^2 - 2(a + b - 2c)y \geq 0$

$\Rightarrow$ $y^2 - 2(a + b - 2c)y + (a - b)^2 \geq 0$

$\Rightarrow$ $[y - (\sqrt{(a-c)} - \sqrt{(b-c)})^2][y - (\sqrt{(a-c)} + \sqrt{(b-c)})^2] \geq 0$

$\therefore$ $y \leq (\sqrt{(a-c)} - \sqrt{(b-c)})^2$

and $y \geq (\sqrt{(a-c)} - \sqrt{(b-c)})^2$

Hence minimum value of y is

$$(\sqrt{(a-c)} - \sqrt{(b-c)})^2.$$

➤ **Problem 23.** If x be real, prove that the expression,

$2(a - x)\left(x + \sqrt{x^2 + b^2}\right)$ cannot exceed $a^2 + b^2$.

➤ ***Solution*** Let $\sqrt{x^2 + b^2} + x = u$ ...(1)

then $\dfrac{1}{\sqrt{x^2 + b^2} + x} = \dfrac{1}{u}$

$$= \frac{\sqrt{x^2 + b^2} - x}{b^2}$$

or $\sqrt{x^2 + b^2} - x = b^2/u$ ...(2)

From (1) & (2), $2x = u + b^2/u$

Let $y = 2(a - x)\left(x + \sqrt{x^2 + b^2}\right)$

$$= \left[2a - \left(u + \frac{b^2}{u}\right)\right] u$$

$y = 2au - u^2 - b^2$ or $y^2 - 2au + y - b^2 = 0$

Since $u \in R$ $\Rightarrow$ $D \geq 0$

$4a^2 - 4(y - b^2) \geq 0$ $\quad a^2 - (y - b^2) \geq 0$

$a^2 + b^2 \geq y$ $\Rightarrow$ $y \leq a^2 + b^2$

➤ **Problem 24.** Find the greatest value of

$$\frac{\left\{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1\right\}}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)} \text{ for } \forall\, x \in R - \{0\}.$$

➤ ***Solution*** Let $y = \dfrac{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}$.

Put $x + \dfrac{1}{x} = t$

$$\Rightarrow \quad y = \frac{t^4 - [(t^2-2)^2 - 2] - 1}{t^2 + t^2 - 2}$$

$$= \frac{t^4 - [t^4 - 4t^2 + 2] - 1}{2(t^2-1)}$$

$$\Rightarrow \quad y = \frac{4t^2 - 3}{2(t^2-1)} \Rightarrow y = \frac{4t^2 - 4 + 1}{2(t^2-1)}$$

$$= \left(2 + \frac{1}{2(t^2-1)}\right).$$

As $t = x + \dfrac{1}{x} \Rightarrow t^2 \geq 4 \Rightarrow t^2 - 1 \geq 3 \Rightarrow \dfrac{1}{t^2-1} \leq \dfrac{1}{3}$.

$\Rightarrow$ Maximum value of y is $2 + \dfrac{1}{3 \times 2} = \dfrac{13}{6}$

➤ **Problem 25.** Solve for integers x, y, z :

$x + y = 1 - z,\ x^3 + y^3 = 1 - z^2.$

➤ ***Solution*** Eliminating z from the given set of equations, we get

$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$

This factors to

$(x + y)\,(x^2 - xy + y^2 + x + y - 2) = 0.$

Case 1. Suppose $x + y = 0$. Then $z = 1$ and $(x, y, z) = (m, -m, 1)$, where m is an integer given one family of solutions.

Case 2. Suppose $x + y \neq 0$. Then we must have $x^2 - xy + y^2 + x + y - 2 = 0$

This can be written in the form

$(2x - y + 1)^2 + 3(y + 1)^2 = 12.$

Here there are two possibilities :

$2x - y + 1 = 0, \qquad y + 1 = \pm 2;$

or $\quad 2x - y + 1 = \pm 3, \qquad y + 1 = \pm 1.$

Analysing all these case we get the following solutions :

(0, 1, 0), (–2, –3, 6), (1, 0, 0),

(0, – 2, 3), (–2, 0, 3), (–3, –2, 6).

➤ **Problem 26.** If $(ax^2 + bx + c)y + (a'x^2 + b'x + c') = 0$, find the condition so that x may be expressed as a rational function of y.

➤ ***Solution*** The given equation can be written as

$$(a + a'y)x^2 + (b + b'y)x + (c + c'y) = 0 \qquad \text{...(1)}$$

Solving equation (1), we have

$$x = \frac{-(b + b'y) \pm \sqrt{(b + b'y)^2 - 4(a + a'y)(c + c'y)}}{2(a + a'y)}$$

For x to be a rational function of y, the expression under the radical sign must be a perfect square. The expression in y will be a perfect square if its discriminant is zero,

$\Rightarrow \quad (bb' - 2ac' - 2a'c)^2 - (b^2 - 4ac)\,(b'^2 - 4a'c') = 0$

$\Rightarrow \quad (ac' - a'c)^2 = (ab' - a'b)\,(bc' - b'c)$

which is the required condition.

➤ **Problem 27.** A quadratic equation with roots α & β where $\alpha\beta = 4$, satisfies the condition,

$$\frac{\alpha}{\alpha - 1} + \frac{\beta}{\beta - 1} = \frac{a^2 - 7}{a^2 - 4}.$$

Find the set of values of 'a' for which $\alpha, \beta \in (1, 4)$.

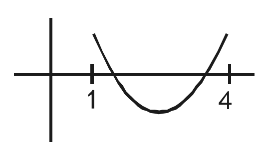
➤ ***Solution*** $\dfrac{\alpha(\beta - 1) + \beta(\alpha - 1)}{(\alpha\beta + 1) - (\alpha + \beta)} = \dfrac{a^2 - 7}{a^2 - 4}$

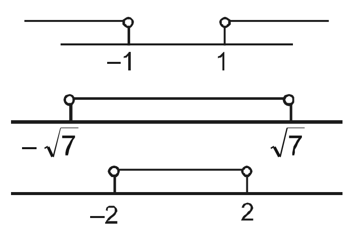
$$\frac{8 - (\alpha + \beta)}{5 - (\alpha + \beta)} = \frac{a^2 - 7}{a^2 - 4}$$

$\Rightarrow \quad \alpha + \beta = a^2 + 1$

Hence quad. equation is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - (a^2 + 1)\,x + 4 = 0$





Let $f(x) = x^2 - (a^2 + 1)\,x + 4 = 0$

$f(1) = 4 - a^2 > 0 \qquad \text{...(1)}$

or $\quad f(4) = 16 - 4a^2 > 0$

$\Rightarrow \quad 4 - a^2 > 0$ ...(2)

Also $1 < \dfrac{a^2+1}{2} < 4$

$\Rightarrow \quad 1 < a^2 < 7$ ...(3)

and $D \geq 0$ ...(4)

From (1), (2), (3) & (4) we get the answer

$$a \in \left(-2, -\sqrt{3}\right] \cup \left[\sqrt{3}, 2\right)$$

➤ **Problem 28.** Find the set of values of 'a' for which the equation, $(1+a)\left(\dfrac{x^2}{x^2+1}\right)^2 - 3a\dfrac{x^2}{x^2+1} + 4a = 0$ have real roots.

➤ ***Solution*** Put $\dfrac{x^2}{1+x^2} = y \quad \Rightarrow \quad y \in [0, 1)$

$(a+1)y^2 - 3ay + 4a = 0$

Find 'a' for which the equation has atleast one solution, for $y = 0$; $a = 0$ ...(1)

Case I : exactly one root between 0 and 1

for $a = -1$; $y = \dfrac{4}{3} \notin (0, 1)$

for $a \neq -1$ equation becomes

$$y^2 - \frac{3a}{a+1}y + \frac{4a}{a+1} = 0$$

(i) $D > 0$ & (ii) $f(0) \cdot f(1) < 0$

$\Rightarrow \quad (-1/2, 0)$ combining with (1)

$\Rightarrow \quad (-1/2, 0]$

Case II : when both roots lie in (0, 1)

(i) $f(a) > 0$; (ii) $f(1) > 0$;

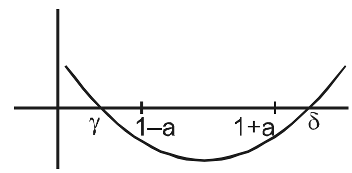
(iii) $D \geq 0$; (iv) $0 < -b/2a < 1$

No solution in this case

Ans. : $-1/2 < x \leq 0$

➤ **Problem 29.** If α, β are the roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation, $x^2 - 2(a+1)x + a(a-1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the values of 'a'.

➤ ***Solution*** $\alpha, \beta = \dfrac{2 \pm \sqrt{4 - 4(a^2-1)}}{2} = 1 + a$ or $1 - a$



Let $f(x) = x^2 - 2(a+1)x + a(a-1)$

For $\alpha, \beta \in (\gamma, \delta)$ we should have

$f(1+a) < 0$ and

$f(1-a) < 0 \quad \Rightarrow \quad a \in \left(-\frac{1}{4}, 1\right)$

➤ **Problem 30.** Prove that $\dfrac{1}{3} \leq \dfrac{\sec^4\theta - 3\tan^2\theta}{\sec^4\theta - \tan^2\theta} < 1$.

➤ ***Solution*** Let $x = \dfrac{\sec^2\theta - 3\tan^2\theta}{\sec^4\theta - \tan^2\theta}$

$\Rightarrow \quad [(1+\tan^2\theta)^2 - \tan^2\theta]x = (1+\tan^2\theta)^2 - 3\tan^2\theta$

$\Rightarrow \quad (x-1)\tan^4\theta + (x+1)\tan^2\theta + (x-1) = 0$ ...(1)

Since $\tan^2\theta$ is real, therefore we have

$(x+1)^2 - 4(x-1) \geq 0$

$\Rightarrow \quad (x+1+2x-2)(x+1-2x+2) \geq 0$

$\Rightarrow. \quad (3x-1)(-x+3) \geq 0$

$\Rightarrow \quad (3x-1)(x-3) \leq 0$

$\Rightarrow \quad \dfrac{1}{3} \leq x \leq 3$ ...(2)

Also, since $\tan^2\theta$ is positive, both the roots of equation (1) must be positive. Thus, we have

sum of the roots > 0

$\Rightarrow \quad -\dfrac{x+1}{x-1} > 0$ ...(3)

and product of the roots > 0 which is true

$\forall\, x \in R - \{1\}$ ...(4)

Intersection of inequalities (2), (3) and (4) gives $x \in [1/3, 1)$ which is the desired result.

➤ **Problem 31.** If p(x) is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have p(a) = b, p(b) = c and p(c) = a.

➤ ***Solution*** Suppose a, b, c are distinct integers such that

p(a) = b, p(b) = c and p(c) = a. Then

$p(a) - p(b) = b - c$,
$p(b) - p(c) = c - a$, $p(x) - p(a) = a - b$.
But for any two integers $m \neq n$, $m - n$ divides $p(m) - p(n)$. Thus we get,

$$a - b \mid b - c, \qquad b - c \mid c - a, \qquad c - a \mid a - b.$$

These force $a = b = c$, a contradiction, Hence there are no integers a, b, and c such that $p(a) = b$, $p(b) = c$ and $p(c) = a$.

➤ **Problem 32.** Prove that if a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

with integral coefficients assumes the value 7 for four integral values of x then it cannot take the value 14 for any integral value of x.

➤ ***Solution*** Let the polynomial P(x) take on the value 7 at the points $x = a$, $x = b$, $x = c$ and $x = d$. Then a, b, c and d are four integral roots of the equation $P(x) - 7 = 0$. This means that the polynomial $P(x) - 7$ is divisible by $x - a$, $x - b$, $x - c$ and $x - d$, that is

$$P(x) - 7 = (x - a)(x - b)(x - c)(x - d)\,p(x)$$

where p(x) may be equal to 1.
Now let us suppose that the polynomial P(x) assumes the value 14 for an integral value $x = A$. On substituting $x = A$ into the last equality we obtain

$$7 = (A - a)(A - b)(A - c)(A - d)\,p(A)$$

which is impossible because the integral numbers $A - a$, $A - b$, $A - c$ and $A - d$ are all distinct and the number 7 cannot be factored into five integers among which at least four are different.

➤ **Problem 33.** Solve in R the equation

$$2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + ... + 2x + 3 = 0.$$

➤ ***Solution*** $2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + ... + 2x + 3$

$$= (2x + 3)(x^{98} + x^{96} + x^{94} + ... + 1),$$

$$= (2x + 3)\,\frac{(x^{100} - 1)}{x^2 - 1}.$$

The equation $x^{100} - 1 = 0$ has only two real roots, namely ± 1 which are not acceptable. Therefore the given equation has only one real root, namely $-3/2$.

➤ **Problem 34.** Prove that the equation $x^6 + 2x^3 + 5 + ax^3 + a = 0$ has atmost two real roots for all values of $a \in R - \{-5\}$.

➤ ***Solution*** The given expression is

$$(x^3 + 1)^2 + a(x^3 + 1) + 4 = 0$$

If discriminant of the above equation is less than zero i.e. $D < 0$
Then we have six complex roots and no real roots.
If $D \geq 0$, $x^3 + 1 = t$, then the equation reduces to $f(t) = t^2 + at + 4 = 0$
we will get two real roots and other roots will be complex except when $t = 1$ is one of the roots
$\Rightarrow \quad f(1) = 0 \Rightarrow a = -5$.

➤ **Problem 35.** If α, β are the roots of equation $x^2 + px + q = 0$ and also of equation $x^{2n} + p^nx^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of equation $x^n + 1 + (x + 1)^n = 0$, then prove that n must be an even integer.

➤ ***Solution*** $\alpha^n + \beta^n = -p^n$ ...(1)

Again since $\frac{\alpha}{\beta}$ is a root of equation

$$x^n + 1 + (x + 1)^n = 0$$

$$\therefore \quad \left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

or $\quad \alpha^n + \beta^n + (\alpha + \beta)^n = 0$

or $\quad \alpha^n + \beta^n = -(\alpha + \beta)^n = -(-p)^n$ ...(2)

From (1) and (2), we have

$$-p^n = -(-p)^n \Rightarrow p^n = (-p)^n$$

$\Rightarrow \quad p^n = (-1)^n p^n \Rightarrow (-1)^n = 1$

$\Rightarrow \quad$ n is an even integer.

➤ **Problem 36.** Solve the equation

$$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$$

➤ ***Solution*** The expression on the left has the factor $x^2 - 1$ corresponding to the roots ± 1. Thus we have

$$6(x^6 - 1) - 25x(x^4 - 1) + 31x^2(x^2 - 1) = 0.$$

Hence the required roots are ± 1 and the roots of

$$6x^4 - 25x^2 + 37x^2 - 25x + 6 = 0.$$

Divide by x^2, then

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0.$$

Put $x + \frac{1}{x} = y$; $\therefore$ $x^2 + \frac{1}{x^2} = y^2 - 2$.

Hence $6y^2 - 25y + 25 = 0$;

$$y = \frac{5}{2} \text{ or } y = \frac{5}{3}.$$

From $x + \frac{1}{x} = \frac{5}{2}$, we have $x = 2$ or $\frac{1}{2}$.

From $x + \frac{1}{x} = \frac{5}{3}$, we have $x = \frac{1}{6}(5 \pm \sqrt{-11})$.

Thus the required roots are $\pm 1, 2, \frac{1}{2}, \frac{1}{6}(5 \pm \sqrt{-11})$.

➤ **Problem 37.** Solve the equation $x^3 - 15x = 126$.

➤ ***Solution*** Put $y + z$ for x, then

$$y^3 + z^3 + (3yz - 15)x = 126 ;$$

Put $3yz - 15 = 0$

then $y^3 + z^3 = 126$;

also $y^3z^3 = 125$,

hence y^3, z^3 are the roots of the equation

$$t^2 - 126t + 125 = 0 ;$$

$\therefore$ $y^3 = 125, z^3 = 1$;

$\therefore$ $y = 5, z = 1$

Thus $y + z = 5 + 1 = 6$;

$$\omega y^{2} + \omega^2 z = -3 + 2\sqrt{-3} \;;$$

$$\omega^2 y + \omega z = -3 - 2\sqrt{-3} \;;$$

and the roots are $6, -3 + 2\sqrt{-3}, -3 - 2\sqrt{-3}$.

STUDY TIP The cube roots of unity are 1, ω, ω^2 where

$$\omega = \frac{-1 \pm \sqrt{-3}}{2}$$

We have $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

The roots of $z^2 + z + 1 = 0$ are called ω, ω^2.

➤ **Problem 38.** Solve the equation $x^4 - 2x^2 + 8x - 3 = 0$

➤ ***Solution*** Assume

$$x^4 - 2x^2 + 8x - 3 = (x^2 + kx + l)(x^2 - kx + m) ;$$

then by equating coefficients, we have

$$l + m - k^2 = -2,\ k(m - l) = 8,\ lm = -3 ;$$

we obtain $(k^3 - 2k + 8)(k^3 - 2k - 8) = -12k^2$,

or $k^6 - 4k^4 + 16k^2 - 64 = 0$.

This equation is clearly satisfied when $k^2 - 4 = 0$, or $k = \pm 2$. It will be sufficient to consider one of the values of k; putting $k = 2$, we have

$$m + l = 2,\ m - l = 4 \text{ ; that is, } l = -1,\ m = 3.$$

Thus $x^4 - 2x^2 + 8x - 3 = (x^2 + 2x - 1)(x^2 - 2x + 3)$;

hence $x^2 + 2x - 1 = 0$, and $x^2 - 2x + 3 = 0$;

and therefore the roots are $-1 \pm \sqrt{2}$, $1 \pm \sqrt{-2}$.

➤ **Problem 39.** If the product of two roots of the equation

$4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0$ is 1, find the roots.

➤ ***Solution*** Suppose the roots are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta = 1$.

Now, $\sigma_1 = (\alpha + \beta) + (\gamma + \delta) = -\frac{-24}{4} = 6$ ....(1)

$$\sigma_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{31}{4}$$

$\Rightarrow$ $(\alpha + \beta)(\gamma + \delta) + \gamma\delta = \frac{31}{4} - 1 = \frac{27}{4}$ ...(2)

$$\sigma_3 = \gamma\delta(\alpha + \beta) + \alpha\beta(\gamma + \delta) = \frac{-3}{2}$$

$\Rightarrow$ $\gamma\delta(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2}$ ...(3)

$\sigma_4 = \alpha\beta\gamma\delta = -2$

$\Rightarrow$ $\gamma\delta = -2$ ...(4)

From Eq. (2) and Eq. (4), we get

$(\alpha + \beta)(\gamma + \delta) = \frac{35}{4}$ ...(5)

From Eq. (3) and Eq. (4), we get

$-2(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2}$ ...(6)

From Eq. (1) and Eq. (6), we get

$$3(\alpha+\beta)=\frac{15}{2} \text{ or } \alpha+\beta=\frac{5}{2} \qquad ...(7)$$

and $\alpha\beta = 1$ ...(Given)

$\Rightarrow \quad \beta = 1/\alpha$

Putting the value of β in Eq. (7), we get

$$a+\frac{1}{\alpha}=\frac{5}{2}$$

$\Rightarrow \quad 2\alpha^2 - 5\alpha + 2 = 0 \Rightarrow (2\alpha - 1)(\alpha - 2) = 0$

$\Rightarrow \quad \alpha = 1/2$ or $\alpha = 2$ and hence $\beta = 2$ or $\beta = 1/2$

Taking $\alpha = 1/2$ and $\beta = 2$ and substituting in Eq. (5),

we get $\quad \gamma+\delta=\frac{7}{2}$

and we know that $\gamma\delta = -2$

Again solving for γ and δ, we get

$$\gamma=\frac{-1}{2} \text{ and } \delta = 4 \text{ or } \delta=\frac{-1}{2} \text{ and } \gamma = 4$$

Thus the four roots are $\frac{1}{2}, \frac{-1}{2}$, 2 and 4.

➤ **Problem 40.** If a, b, c are the roots of the equation

$$x^3 + px^2 + qx + r = 0,$$

form the equation whose roots are

$$a-\frac{1}{bc},\ b-\frac{1}{ca},\ c-\frac{1}{ab}$$

➤ ***Solution*** When x = a in the given equation $y = a - \frac{1}{bc}$ in the transformed equation ; but

$$a-\frac{1}{bc}=a-\frac{a}{abc}=a+\frac{a}{r}\ ;$$

and therefore the transformed equation will be obtained by the substitution

$$y = x+\frac{x}{r}, \text{ or } x=\frac{ry}{1+r}\ ;$$

thus the required equation is

$$r^2y^3 + pr(1 + r)y^2 + q(1 + r)^2y + (1 + r)^3 = 0$$

➤ **Problem 41.** Form the equation whose roots are the squares of the differences of the roots of the cubic

$$x^3 + qx + r = 0$$

➤ ***Solution*** Let a, b, c be the roots of the cubic ; then the roots of the required equation are

$$(b - c)^2, (c - a)^2, (a - b)^2.$$

Now

$$(b-c)^2 = b^2 + c^2 - 2bc = a^2 + b^2 + c^2 - a^2 - \frac{2abc}{a}$$

$$= (a + b + c)^2 - 2(bc + ca + ab) - a^2 - \frac{2abc}{a}$$

$$= -2q - a^2 + \frac{2r}{a}\ ;$$

also when x = a in the given equation, $y = (b - c)^2$ in the transformed equation ;

$$\therefore \quad y = -2q - x^2 + \frac{2r}{x}$$

Thus we have to eliminate x between the equations

$$x^3 + qx + r = 0,$$

and $\quad x^3 + (2q + y)x - 2r = 0$

By subtraction $(q + y)x = 3r$; or $x=\frac{3r}{q+y}$

Substituting and reducing, we obtain

$$y^3 + 6qy^2 + 9q^2y + 27yr^2 + 4q^3 = 0$$

Conclusion : If a, b, c are real, $(b-c)^2, (c-a)^2, (a-b)^2$ are all positive, therefore $27r^2 + 4q^3$ is negative.

Hence in order that the equation $x^3 + qx + r = 0$ may have all its roots real $27r^2 + 4q^3$ must be negative.

If $27r^2 + 4q^3 = 0$, the transformed equation has one root zero, therefore the original equation has two equal roots.

If $27r^2 + 4q^3$ is positive, the transformed equation has a negative root, therefore the original equation must have two imaginary roots, since it is only such a pair of roots which can produce a negative root in the transformed equation.

➤ **Problem 42.** Given that the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

has four positive roots, prove that

(a) $pr - 16s \geq 0$ (b) $q^2 - 36s \geq 0$

with equality in each case holding if and only if the four roots are equal.

➤ ***Solution*** Let the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

be $\alpha, \beta, \gamma, \delta$, so that $\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$

Now $\Sigma\alpha = -p$,

$\Sigma\alpha\beta = q$,

$\Sigma\alpha\beta\gamma = -r$,

$\alpha\beta\gamma\delta = s$,

(i) $pr = \Sigma\alpha \,.\, \Sigma\alpha\beta\gamma$.

By A.M.- G.M. inequality

$$\frac{1}{4}\Sigma\alpha \geq (\alpha\beta\gamma\delta)^{1/4}, \quad ...(1)$$

$$\frac{1}{4}\Sigma\alpha\beta\gamma \geq (\alpha\beta\gamma\delta)^{3/4}, \quad ...(2)$$

By multiplying corresponding sides of the above inequalities, we have

$$\frac{1}{16}\Sigma\alpha\,\Sigma\alpha\beta\gamma \geq \alpha\beta\gamma\delta,$$

i.e., $pr - 16s \geq 0$. ...(3)

Equality holds in (3) $\Leftrightarrow$ equalities hold in both (1) and (2) $\Leftrightarrow \alpha, \beta, \gamma, \delta$ are all equal, and $\alpha\beta\gamma, \alpha\beta\delta, \alpha\gamma\delta$ are all equal $\Leftrightarrow \alpha = \beta = \gamma = \delta$.

(ii) $q^2 = (\Sigma\alpha\beta)^2 \geq [\,6\,(\alpha\beta \,.\, \alpha\gamma \,.\, \alpha\delta \,.\, \beta\gamma \,.\, \beta\delta \,.\, \gamma\delta\,)^{1/6}\,]^2$,

$= 36\alpha\beta\gamma\delta$,

$= 36s$.

Therefore $q^2 \geq 36s$.

Equality holds if and only if $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$ are all equal,

i.e., if and only if $\alpha = \beta = \gamma = \delta$.

STUDY TIP Let $a_1, a_2, a_3, \ldots\ldots a_n$ be n positive real numbers, then we define their

Arithmetic Mean A.M. = $\dfrac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}$ and

Geometric Mean G.M. = $(a_1 a_2 a_3 \ldots\ldots\ldots\ldots a_n)^{1/n}$

It can be shown that A.M. $\geq$ G.M. and equality holds places iff $a_1 = a_2 = a_3 = \ldots\ldots\ldots\ldots\ldots = a_n$.

➤ **Problem 43.** Find the number and position of the real roots of the equation

$$x^6 - 5x^5 - 7x^2 + 8x + 20 = 0.$$

➤ ***Solution*** By Descartes' Rule of Signs we see that there cannot be more than two positive roots and there cannot be more than two negative roots.

Now $f(1) > 0$, $f(2) < 0$; thus one real root lies between 1 and 2. Since $f(\infty)$ is ∞, there must be another positive root which is found to lie between 5 and 6.

Change x into –x, then the negative roots of the given equation are positive roots of

$$x^6 + 5x^5 - 7x^2 - 8x + 20 = 0.$$

Now f(x) must clearly be positive for all positive values between 0 and 1; and if $x > 1$.

$$f(x) > 6x^4 - 15x^2 + 20,$$

which is always positive since

$$4 \times 6 \times 20 - 15^2 > 0$$

Hence there can be no real negative roots.

Things to Remember

1. **The quadratic formula**

   The solution of the quadratic equation , $ax^2 + bx + c = 0$ is given by

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. **Sum and Product of roots**

   Vieta's Theorem : If a quadratic equation $ax^2 + bx + c = 0$ has roots α & β then their sum is equal to $-\dfrac{b}{a}$ while their product is $\dfrac{c}{a}$

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then we have
$ax^2 + bx + c = a(x - \alpha)(x - \beta)$

3. An **identity** in x is satisfied by all permissible values of x, where as an **equation** in x is satisfied by some particular value of x.

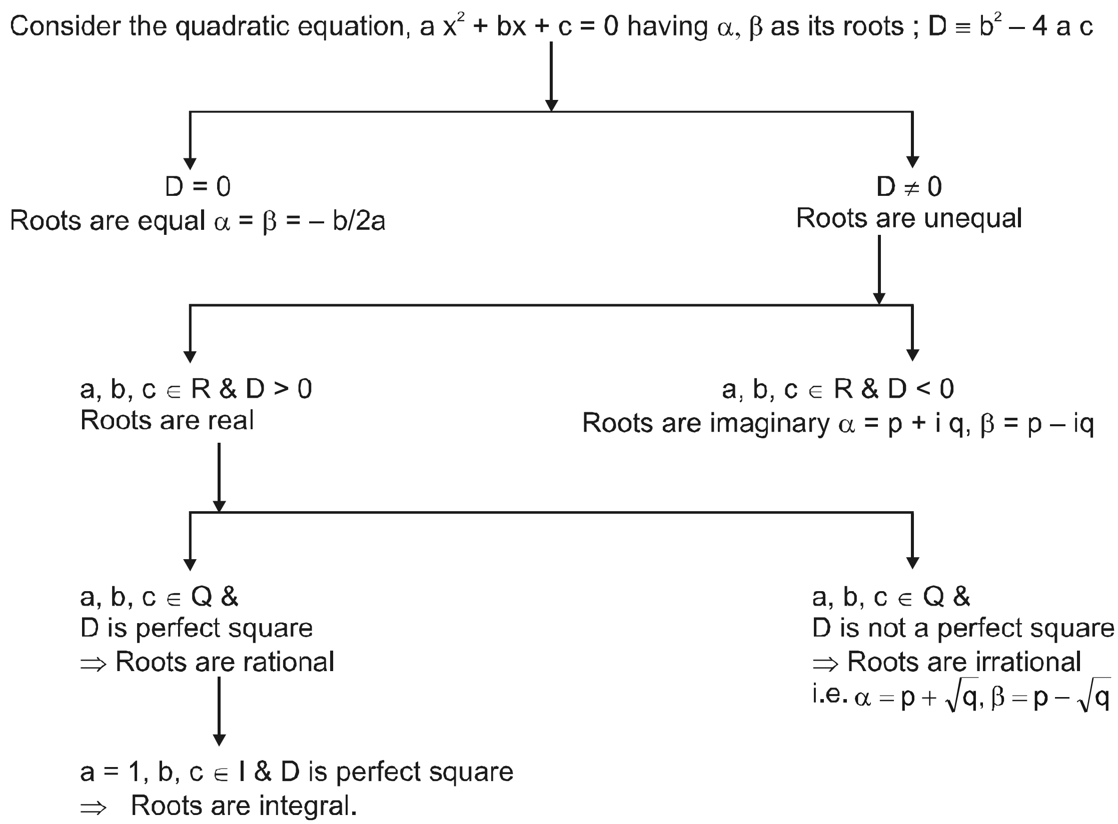
If a quadratic equation is satisfied by three or more distinct values of 'x', then it is an identity.

Two equations in x are **identical** if and only if the coefficient of similar powers of x in the two equations are proportional. Identical equations have the same roots.

4. Newton's Theorem :

If α, β are roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then for $n > 2$, $n \in N$, we have $aS_n + bS_{n-1} + cS_{n-2} = 0$

5. Nature of Roots



6. Common Roots of Quadratic Equations

Two Common Roots : $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have both roots common

$$\text{if } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Atleast one Common Root : Let α be a common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$

$$\text{Then } \frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

The condition for atleast one common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

If $f(x) = 0$ & $g(x) = 0$ are two polynomial equations having some common roots(s) then those common root(s) is/are also the root(s) of $h(x) = a\,f(x) + b\,g(x) = 0$.

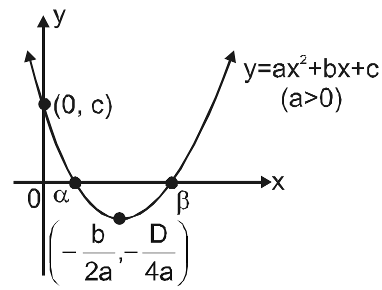
7. Graph of a quadratic function

$$y = f(x) = a\,x^2 + b\,x + c$$

$$\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

(i) The graph between x, y is always a parabola.

(ii) If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.



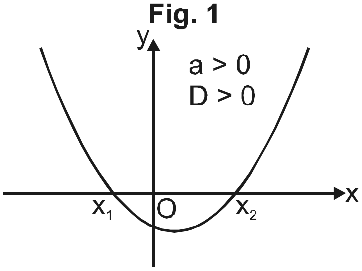
(iii) the co-ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(iv) the parabola intersect the y–axis at point (0, c)

(v) the x co-ordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation $f(x) = 0$. Hence the parabola may or may not intersect the x–axis at real points.

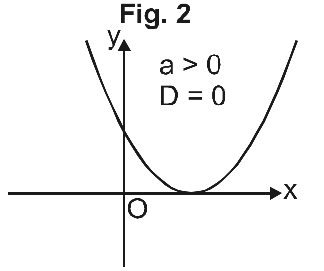
8. Position of graph based on a and D

Fig. 1



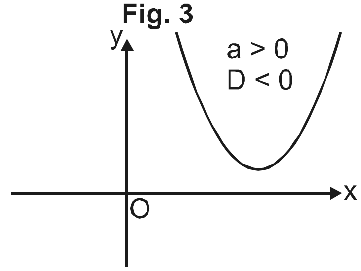
Roots are real & distinct

Fig. 2



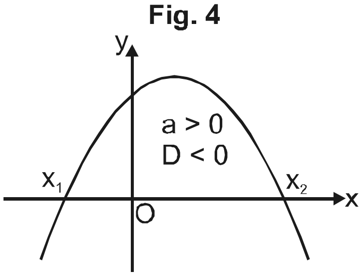
Roots are coincident

Fig. 3



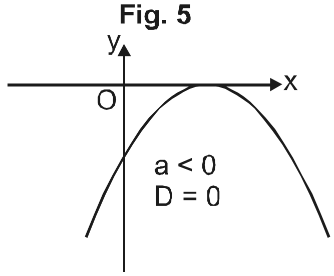
Roots are complex conjugate

Fig. 4



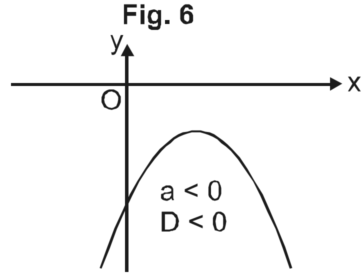
Roots are real & distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

9. Quadratic inequality

All the relevant properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ are given in Table :

	a	D	Root of $f(x) = 0$	Sign of f(x)
1.	$a > 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x)$ > 0 for $x < \alpha$ < 0 for $\alpha < x < \beta$ > 0 for $x > \beta$
2.	$a > 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x)$ > 0 for $x < \alpha$ $= 0$ for $x = \alpha$ > 0 for $x > \alpha$
3.	$a > 0$	$D < 0$	Imaginary roots α, β	$f(x)$ > 0 for all x
4.	$a < 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x)$ < 0 for $x < \alpha$ > 0 for $\alpha < x < \beta$ < 0 for $x > \beta$
5.	$a < 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x)$ < 0 for $x < \alpha$ $= 0$ for $x = \alpha$ < 0 for $x > \alpha$
6.	$a < 0$	$D < 0$	Imaginary roots α, β	$f(x)$ < 0 for all x

10. Range of Quadratic Expression $f(x) = ax^2 + bx + c$

(i) Range when $x \in R$

If $a > 0 \Rightarrow \quad f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \Rightarrow \quad f(x) \in \left[-\infty, \frac{D}{4a}\right)$

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

(ii) Range in restricted domain

Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$f(x) \in \lfloor \min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\} \rfloor$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

11. Resolution of a Second Degree Expression in X and Y

The condition that a quadratic function of x and y may be resolved into two linear factors is that

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

OR $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

and it can be written as a product of two linear factors with real coefficients if apart from the above condition we have either (i) $h^2 - ab > 0$, or (ii) $h^2 - ab = 0$, $hg - af = 0$, $g^2 - ac \geq 0$

12. Location of roots

Necessary and Sufficient Conditions

1. roots are of opposite sign $\frac{c}{a} < 0$ $\quad \alpha < 0, \beta > 0$
2. roots are of same sign

   $D \geq 0$ and $\frac{c}{a} > 0$ $\quad \alpha < 0, \beta < 0$ or $\alpha > 0, \beta > 0$
3. both roots are positive

   $D \geq 0, -\frac{b}{a} > 0$ and $\frac{c}{a} > 0$ $\quad \alpha > 0, \beta > 0$
4. both roots are negative

   $D \geq 0, -\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ $\quad \alpha < 0, \beta < 0$
5. both roots greater than k

   $a.f(k) > 0, -\frac{b}{2a} > k$, and $D \geq 0$ $\quad \alpha > k, \beta > k$
6. both roots are less than k

   $a.f(k) > 0, -\frac{b}{2a} < k$, and $D \geq 0$ $\quad \alpha < k, \beta < k$
7. k lies between the roots

   $a.f(k) < 0$ $\quad \alpha < k, \beta > k$
8. one root less than d and other greater than e

   $a.f(d) < 0,\ a.f(e) < 0$ $\quad \alpha < d, \beta > e$
9. both roots are lie in (d , e)

   $a.f(d) > 0, a.f(e) > 0, d < -\frac{b}{2a} < e$, and $D \geq 0$

   $\alpha < k, \beta < k$
10. exactly one root lies in (d , e)

    $f(d).f(e) < 0^*$ $\quad \alpha < k, \beta < k$

    * Refer theory since more conditions are required.

13. Theory of Equations

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_n$ are the roots of the equation;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots. + a_{n-1}x + a_n = 0$ where $a_0, a_1, \ldots. a_n$ are all real & $a_0 \neq 0$ then,

$$\Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ldots.., \alpha_1\alpha_2\alpha_3\ldots\ldots..\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Note

(i) Remainder Theorem : Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x) is divided $(x - a)$, then the remainder is equal to P(a).

Factor Theorem : Let P(x) be polynomial of degree greater than of equal to 1 and 'a' be a real number such that $P(a) = 0$, then $(x - a)$ is a factor of P(x). Conversely, if $(x - a)$ is a factor of P(x), then $P(a) = 0$.

(ii) Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity .

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs**.

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ and β is not a perfect square. Also if $\sqrt{p}$ and $\sqrt{q}$ are two dissimilar quadratic surds and $\sqrt{p} + \sqrt{q}$ is a root then $\pm\sqrt{p} \pm \sqrt{q}$ are also roots of the equation.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(vi) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

(vii) If α is an r-multiple root of $f(x) = 0$, it is then an $(r-1)$ multiple root of $f'(x) = 0$, an $(r-2)$ multiple root of $f''(x) = 0$, and so on.

(viii) Descartes' rule of signs :

(a) The maximum number of positive roots of f(x) is equal to the number of sign changes in the coefficients of f(x). Let this number be 'p'.

(b) The maximum number of negative roots of f(x) is equal to the number of sign changes in the coefficients of f(–x). Let this number be 'q'.

(c) The minimum number of imaginary roots of f(x) is equal to $n - (p + q)$.

(ix) Let f(x) be a polynomial having roots α and β where $\alpha < \beta$, then there exists atleast one number $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$.

[Objective Exercises]

Single Correct Answer Type

1. The value of x satisfying the equation

$$\frac{6x + 2a + 3b + c}{6x + 2a - 3b - c} = \frac{2x + 6a + b + 3c}{2x + 6a - b - 3c}$$ is

(A) $x = ab/c$ (B) $2ab/c$
(C) $ab/3c$ (D) $ab/2c$

2. If $a < c < b$ then the roots of the equation $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$ are
(A) imaginary (B) real
(C) one real and one imaginary
(D) equal and imaginary

3. If the ratio of the roots of $x^2 + bx + c = 0$, is same as that of $x^2 + qx + r = 0$, then
(A) $r^2b = qc^2$ (B) $r^2c = qb^2$
(C) $c^2r = q^2b$ (D) $b^2r = q^2c$

4. Let α, β be the roots of the equation $ax^2 + 2bx + c = 0$ and γ, δ be the roots of the equation $px^2 + 2qx + r = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P. then
(A) $q^2ac = b^2pr$ (B) $qac = bpr$
(C) $c^2 pq = r^2ab$ (D) $p^2ab = a^2qr$

5. If $a > b > 0$, then the value of

$$\sqrt{ab+(a-b)\sqrt{ab+(a-b)\sqrt{ab+(a-b)\sqrt{ab+...}}}}$$

is
(A) independent of only b
(B) independent of only a
(C) independent of both a & b
(D) dependent on both a & b .

6. If both the roots of the equations
$$k(6x^2 + 3) + rx + 2x^2 - 1 = 0$$
and $$6k(2x^2 + 1) + px + 4x^2 - 2 = 0$$
are common, then $2r - p$ is equal to
(A) 1 (B) –1 (C) 2 (D) 0

7. If the quadratic equations,
$ax^2 + 2cx + b = 0$ & $ax^2 + 2bx + c = 0$ $(b \neq c)$ have a common, root, then $a + 4b + 4c$ is equal to
(A) – 2 (B) 2 (C) 0 (D) 1

8. If α, β be the roots of $x^2 - a(x - 1) - b = 0$, then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} + \frac{2}{a+b}$ is

(A) $\frac{4}{(a+b)}$ (B) $\frac{1}{(a+b)}$

(C) 0 (D) $\frac{2}{(a+b)}$

9. If $a, c \in Q$ and the roots of the equation $cx^2 + (2 + \sqrt{2})x + 2a\left(1 + \frac{1}{\sqrt{2}}\right) = 0$ are real and distinct, then the roots of the equation $x^2 - 2cax + 1 = 0$ will be

(A) integer (B) rational
(C) irrational (D) imaginary

10. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the quadratic equation whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ is
(A) $(a - b + c)x^2 + (b - 2c)x + c = 0$
(B) $(a - b + c)x^2 - (b - 2c)x + c = 0$
(C) $(a - b + c)x^2 + (b - 2c) - c = 0$
(D) None of these

11. The greatest value of $\frac{4}{4x^2 + 4x + 9}$ is
(A) 4/9 (B) 4 (C) 9/4 (D) 1/2

12. If roots of the equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ are of opposite signs, then a lies in the interval
(A) $(-\infty, 1)$ (B) $(-\infty, 0)$
(C) $(1, 2)$ (D) $(3/2, 2)$

13. The number of integral values of a for which $x^2 - (a - 1)x + 3 = 0$ has both roots positive and $x^2 + 3x + 6 - a = 0$ has both roots negative is
(A) 0 (B) 1
(C) 2 (D) infinite

14. If the two roots of the equation, $x^3 - px^2 - qx - r = 0$ are equal in magnitude but opposite in sign then
(A) $pr = q$ (B) $qr = p$
(C) $pq = r$ (D) None

15. If α, β & γ are the roots of the equation, $x^3 - x - 1 = 0$ then the value of $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$ is
(A) zero (B) – 1 (C) – 7 (D) 1

16. If roots of the equation $x^2 - bx + c = 0$ are two successive integers, then $b^2 - 4c$ equal
(A) 1 (B) 2 (C) 3 (D) 4

17. If the equation $\frac{a}{x-a} + \frac{b}{x-b} = 1$ has roots equal in magnitude but opposite in sign, then the value of a + b is
(A) – 1 (B) 0
(C) 1 (D) None of these

18. The number of solutions of the equation $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4$ is
(A) zero (B) one
(C) two (D) more than two

19. The set of values of 'p' for which the expression $x^2 - 2px + 3p + 4$ is negative for atleast one real x is
(A) ϕ (B) $(-1, 4)$
(C) $(-\infty, -1) \cup (4, \infty)$ (D) $\{-1, 4\}$

20. If one root of the equation $x^2 + bx + a = 0$ & $x^2 + ax + b = 0$ is common and $a \neq b$ then
(A) $a + b = 0$ (B) $a + b = -1$
(C) $a - b = 1$ (D) $a + b = 1$

21. If α_1, α_2, & β_1, β_2, are roots of the equations $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively and a non- zero solution of the system of equations $\alpha_1 y + \alpha_2 z = 0$, $\beta_1 y + \beta_2 z = 0$ exist then
(A) $p^2br = a^2qc$ (B) $b^2pr = q^2ac$
(C) $r^2pb = c^2 ar$ (D) None of these

22. If α, β are the roots of the equation $ax^2 + 3x + 2 = 0$ $(a > 0)$, then $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ is greater than
(A) 0 (B) 1
(C) 2 (D) none

23. If $(\lambda^2 + \lambda - 2)x^2 + (\lambda + 2)x < 1$, $x \in R$, then λ belongs to the interval
(A) $(-2, 1)$ (B) $(-2, 2/5)$
(C) $(2/5, 1)$ (D) None of these

24. The number of positive integral values of k for which $(16x^2 + 12x + 39) + k(9x^2 - 2x + 11)$ is a perfect square is

(A) 2 (B) 0
(C) 1 (D) None

25. If x is real, then the least value the expression $\frac{x^2 - 6x + 5}{x^2 + 2x + 1}$ is

(A) -1 (B) $-1/2$
(C) $-1/3$ (D) none

26. If x is real, then $\frac{x^2 + 2x + c}{x^2 + 4x + 3c}$ can take all real values if

(A) $0 < c < 2$ (B) $0 < c < 1$
(C) $0 \le c \le 1$ (D) none

27. If the roots of $x^2 - p(x + 1) + c = 0$ are α & β then the value of $\alpha^2 + (2p)\alpha\beta + \beta^2$ is

(A) pc (B) $p^2 + pc$
(C) $p^2 - pc$ (D) $-pc$

28. The value(s) of 'b' for which the equation, $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is/are

(A) $b = -12$
(B) $b = 4$
(C) $b = 4$ or $b = -12$
(D) $b = -4$ or $b = 12$

29. If the quadratic equation, $ax^2 + bx + a^2 + b^2 + c^2 - ab - bc - ca = 0$ where a, b, c are distinct reals, has imaginary roots then

(A) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 > 0$
(B) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 < 0$
(C) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$
(D) none of these

30. Three roots of the equation, $x^4 - px^3 + qx^2 - rx + s = 0$ are tan A, tan B & tan C where A, B, C are the angles of a triangle . The fourth root of the equation is

(A) $\frac{p-r}{1-q+s}$ (B) $\frac{p-r}{1+q-s}$
(C) $\frac{p+r}{1-q+s}$ (D) $\frac{p+r}{1+q-s}$

31. Number of values of 'p' for which the equation, $(p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than two roots, is

(A) 0 (B) 1
(C) 2 (D) none

32. The roots of the equation $(x - a)(x - b) = a^2 - 2b^2$ are real and distinct for all $a > 0$, provided

(A) $-a \le b < \frac{5}{7}a$ (B) $-a < b < \frac{5}{7}a$
(C) $-a < b \le \frac{5}{7}a$ (D) $-2a < b < \frac{7}{5}a$

33. If the roots of the quadratic equation $x^2 + 6x + b = 0$ are real and distinct and they differ by atmost 4 then the range of values of b is

(A) $[-3, 5]$ (B) $[5, 9)$
(C) $[6, 10]$ (D) none

34. If the equation $\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$ has a solution then k must lie in the interval

(A) $(-4, -2)$ (B) $[-3, 2)$
(C) $(-4, -3)$ (D) $[-3, -2]$

35. For non-zero real distinct a, b, c, the equation, $(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$ has non-zero real roots. One of these roots is also the root of the equation

(A) $a^2x^2 - a(b - c)x + bc = 0$
(B) $a^2x^2 + a(c - b)x - bc = 0$
(C) $(b^2 + c^2)x^2 - 2a(b + c)x + a^2 = 0$
(D) $(b^2 - c^2)a^2 + 2a(b - c)x - a^2 = 0$

36. The values of a, b, c such that $2x^2 - 5x - 1 \equiv a(x+1)(x-2) + b(x-2)(x-1) + c(x-1)(x+1)$ are respectively :

(A) –2, 1, –1 (B) 2, 1, –1
(C) 1, 2, –1 (D) None of these

37. If a + b + c = 0, then the roots of the equations $3ax^2 + 4bx + 5c = 0$ are
(A) Positive (B) Negative
(C) Real and distinct (D) Imaginary

38. The equation $(x - a)^3 + (x - b)^3 + (x - c)^3$ has
(A) three distinct real roots
(B) one real and two imaginary roots.
(C) three real roots, two of which are equal
(D) None of these

39. The equation $x^5 - 5ax + 4b = 0$ has
(A) three real roots if $a^5 > b^4$
(B) only one real root if $a^5 > b^4$
(C) five real roots
(D) None of these

40. The value of 'a' for which the equation $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$, have a common root is
(A) a = 2 (B) a = –2
(C) a = 0 (D) None

41. The necessary and sufficient condition for the equation $(1 - a^2)x^2 + 2ax - 1 = 0$ to have roots lying in the interval (0, 1) is
(A) a > 0 (B) a < 0
(C) a > 2 (D) None

42. If α and β are the roots of $ax^2 + bx + c = 0$, then the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ has roots.
(A) $\frac{\alpha}{1-\alpha}, \frac{\beta}{1-\beta}$ (B) $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$
(C) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (D) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

43. If one root of the equation $ix^2 - 2(1 + i)x + 2 - i = 0$ is (3 – i), then the other root is
(A) 3 + i (B) 3 + 2i
(C) –1 + i (D*) –1 –i

44. The equation
$$\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\gamma} = m - n^2x$$ (a, b, c, m, n ∈ R) has necessarily
(A) all the roots real
(B) all the roots imaginary
(C) two real and two imaginary roots
(D) two rational and two irrational roots.

45. The equation $x^5 - 2x^2 + 7 = 0$ has
(A) atleast two imaginary roots
(B) two negative real roots
(C) three positive real roots
(D) None of these

46. If the equation $ax^3 - 9x^2 + 12x - 5 = 0$ has two equal real roots then the 'a' equals
(A) 2 (B) $\frac{25}{54}$
(C) 1 (D) None

47. If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$; a ≠ c have a negative common root then the value of (a – b + c) is
(A) 0 (B) 2
(C) 1 (D) None

48. The equation $ax^2 + bx + a = 0$, $x^3 - 2x^2 + 2x - 1 = 0$ have two roots common. Then a + b must be equal to
(A) 1 (B) –1
(C) 0 (D) None

49. The equation $(x - 3)^9 + (x - 3^2)^9 + ... + (x - 3^9)^9 = 0$ has
(A) all the roots real
(B) one real & 8 imaginary roots
(C) real roots namely x = 3, 3^2, ..., 3^9
(D) None of these

50. If $f(x) = \sum_{r=0}^{100} a_r x^r$ and f(0) and f(1) are even numbers, then for any integer x
(A) f(x) is odd or even according as x is odd or even
(B) f(x) is even or odd according as x is odd or even
(C) f(x) is even for all integral values of x
(D) f(x) is odd for all integral values of x.

Multiple Correct Answer Type for JEE Advanced

51. The quadratic expression $(ax+b)(x-2)+c(x^2+3)$ is equal to 14 for all values of x. Then
(A) $a=-2, b=-4, c=2$
(B) $a=2, b=4, c=-2$
(C) $a=2, b=-4, c=2$
(D) None of these

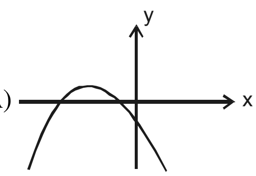
52. The quadratic equation $ax^2 + bx + c = 0$ has real roots if :
(A) $a < -1,\ 0 < c < 1,\ b > 0$
(B) $a < -1,\ -1 < c < 0,\ 0 < b < 1$
(C) $a < -1,\ c < 0,\ b > 1$
(D) none

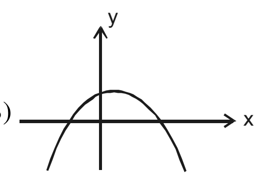
53. The equation $x^2 - 4x + a\sin\alpha = 0$ has real roots
(A) for all values of x
(B) for all values of 'a' provided $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$
(C) for all values of $a \ge 4$ provided $\pi \le \alpha \le 2\pi$
(D) for all a provided $|a| \le 4$

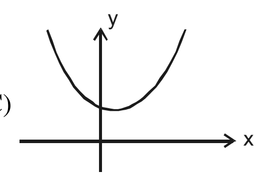
54. The integral values of m for which the roots of the equation $mx^2 + (2m-1)x + (m-2) = 0$ are rational are given by
(A) 15 (B) 12 (C) 6 (D) 4

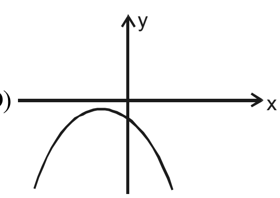
55. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$. Then $2a\alpha + \Delta$ and $2a\beta - \Delta$ can be the roots of the equation:
(A) $x^2 + 2bx + b^2 = 0$
(B) $x^2 - 2bx + b^2 = 0$
(C) $x^2 + 2bx - 3b^2 + 16ac = 0$
(D) $x^2 - 2bx - 3b^2 + 16ac = 0$

56. For which of the following graphs of the quadratic expression $y = ax^2 + bx + c$, then product a b c is negative.

(A)


(B)


(C)


(D)


57. The values of k for which the equation, $x^2 + 2(k-1)x + k + 5 = 0$ has atleast one positive root, are
(A) $[4, \infty)$ (B) $(-\infty, -1] \cup [4, \infty)$
(C) $[-4, -1]$ (D) $(-\infty, -1]$

58. If $p \ne q$ then the equations, $x^2 - px + q = 0$ & $x^2 - qx + p = 0$ are such that :
(A) both can simultaneously have a zero root
(B) if the roots of the first are both positive so are the roots of the second
(C) if the roots of the first are both negative, those of the second have opposite signs

(D) both can have roots belonging to the set {1, 2, 3, 5}

59. Given the equation $(a - x)(b - x) - h^2 = 0$, $a < b$, the incorrect statement is
(A) both the roots are less than b
(B) both the roots lie between a and b
(C) the roots are separated by a and by b.
(D) None of these

60. Which of the following is/are false?
(A) $2x^3 + 7x^2 - 5x + 2x^{-1}$ is a polynomial in standard form of degree 3 and contains four terms.
(B) If two polynomials of degree 3 are added, their sum must be a polynomial of degree 3.
(C) There is no such thing as a polynomial in x having four terms, written in descending powers, and lacking a third-degree term.
(D) Suppose a square garden has area represented by $9x^2$ square metre. If one side is made 7 metre longer and the other side is made 2 metre shorter, then the trinomial that represents the area of the larger garden in $9x^2 + 15x - 14$ square metre.

61. Let a, b, c be the roots of $x^3 - 9x^2 + 11 - 1 = 0$ and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$ and $t = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$, then
(A) $t^2 = 11 + 2s$
(B) $s^4 = 125 + 36t + 8s$
(C) $s^4 - 18s^2 - 8s = -37$
(D) $s^4 - 18s^2 + 8s = 37$

62. Let $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ have roots $a_k + i\, b_k$ for k = 1, 2, 3, 4 where a_k, b_k are all integers. Then
(A) $(a_1^2 + b_1^2)(a_3^2 + b_3^2) = 65$
(B) $a_1 + a_3 = 3$
(C) $|b_1| + |b_2| + |b_3| + |b_4| = 10$
(D) $a_1^2 + b_3^2 = 12$

63. Let $P(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_1, a_2, a_3 \in I$ and $a_1 + a_2 + a_3$ is an even number, then
(A) P(–1) is an even number
(B) P(–1) is an odd number
(C) P(x) = 0 has no integral solutions
(D) P(x) = 0 has atleast one integral solution

64. The equation $x^3 - \frac{3}{4}x = -\frac{\sqrt{3}}{8}$ is satisfied by
(A) $x = \cos\left(\frac{5\pi}{18}\right)$ (B) $x = \cos\left(\frac{7\pi}{18}\right)$
(C) $x = \cos\left(\frac{17\pi}{18}\right)$ (D) $x = \cos\left(\frac{23\pi}{18}\right)$

65. The correct statements about the cubic equation $2x^3 + x^2 + 3x - 2 = 0$ are
(A) it has exactly one positive root
(B) it has either one or three negative roots
(C) it has a root between 0 and 1
(D) it has no imaginary roots

66. The integral value(s) of a for which the equation $x^2 - 2(a - 1)x + (2a + 1) = 0$ has both roots positive is/ are
(A) 3 (B) 4 (C) 1 (D) 5

67. If $b^2 \ge 4ac$ for the equation $ax^4 + bx^2 + c = 0$ then all roots of the equation will be real when
(A) $b > 0, a < 0, c > 0$ (B) $b < 0, a > 0, c > 0$
(C) $b > 0, a > 0, c > 0$ (D) $b > 0, a < 0, c < 0$

68. If $x^2 + px + 1 = 0$ and $(a - b)x^2 + (b - c)x + (c - a) = 0$ have both roots common, then which of the following is true ?
(A) $p = -2$ (B) b, a, c are in A.P.
(C) $2a - 3b + c = 1$ (D) $2c - 3b + c = 0$

69. The equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ has no real root if r is equal to
(A) 8 (B) 20
(C) 23 (D) None

70. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with this cubic, then the possible cases are

(A) $a = b = 0$

(B) a, b are roots of $x^2 + x + 2 = 0$

(C) $a = b = 3$

(D) None of these

Comprehension - 1

Let P(x) be a polynomial such that $P(1) = 1$ and $\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$ for all real x for which both sides are defined.

71. The degree of P(x) is

(A) 2 (B) 3 (C) 4 (D) 5

72. The value of $P(-1)$ is

(A) $\frac{2}{7}$ (B) $-\frac{4}{21}$

(C) $-\frac{5}{21}$ (D) $-\frac{11}{21}$

73. The number of real roots of $P(x) = 0$ is

(A) 5 (B) 3

(C) 2 (D) None

Comprehension - 2

Consider a polynomial equation $x^4 - 63x^3 + ax^2 - 7! = 0$ whose roots are four distinct natural numbers of which three are odd.

74. The largest of the roots is

(A) 16 (B) 48 (C) 24 (D) 32

75. The value of a is

(A) 791 (B) 971

(C) 719 (D) None

76. The number of roots which are prime is

(A) 1 (B) 2

(C) 3 (D) None

Comprehension - 3

Consider the equation $x^4 + ax^3 - bx^2 + ax + 1 = 0$. After dividing the equation by x^2, we can rearrange it as $\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) - b - 2 = 0$. By subtitution, we can solve the equation as a quadratic equation. Let S be the set of points (a, b) in the a – b plane.

77. The equation has four real solutions if and only if

(A) $2a + b - 2 > 0$ and $2a - b + 2 > 0$

(B) $2a + b - 2 < 0$ and $2a - b + 2 < 0$

(C) $2a + b - 2 > 0$ and $2a - b + 2 < 0$

(D) None of these

78. If $b = 2$, then the equation has

(A) four real solutions

(B) two real solutions

(C) no real solution

(D) None of these

79. If $0 \le a, b \le 1$, then for the equation to have atleast one real root, the area of the region S is

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{4}$ (D) None

Comprehension - 4

Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$ and the roots of g are square of the roots of f.

80. The equation $f(x) = 0$ has

(A) atleast one positive root

(B) atleast two negative roots

(C) exactly one negative root

(D) None of these

81. The polynomial $g(x^2)$ is identical with

(A) $f(x^2)$ (B) $(f(x))^2$

(C) $2f(x)\,f(-x)$ (D) $-f(x)\,f(-x)$

82. The value of g(9) is

(A) 889 (B) 899

(C) 961 (D) None

Comprehension - 5

A monic ineducible polynomial with integral coefficients is a polynomial with leading coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 is the factorization into monic irreducible polynomials. Consider two polynomials $f(x) = x^8 + x^4 + 1$ and $g(x) = x^8 + x + 1$.

83. The number of real roots of the equation $f(x) . g(x) = 0$ is

(A) 2 (B) 4
(C) 0 (D) None

84. The highest common factor between f(x) and g(x) is of degree

(A) 1 (B*) 2
(C) 4 (D) None

85. The number of distinct monic irreducible polynomials in the prime factorization of f(x) . g(x) is

(A) 2 (B) 3
(C) 4 (D) None

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. **Assertion (A) :** Let $f(x) = ax^2 + bx + c$, a, b, c ∈ R. If f(x) assumes real values for real values of x and non-real values for non-real values of x, then a = 0.

Reason (R) : If a, b, c are complex numbers, $a \neq 0$, then $\alpha + i\beta$, $\beta \neq 0$ is a root of $ax^2 + bx + c = 0$ if and only if $\alpha - i\beta$ is a root of $ax^2 + bx + c = 0$.

87. **Assertion (A) :** If a, b, c, ∈ Q and $2^{1/3}$ satisfies $a + bx + cx^2 = 0$, then a = 0, b = 0, c = 0.

Reason (R) : A polynomial equation with rational coefficients cannot have irrational roots.

88. **Assertion (A) :** The sum of absolute value of the roots of

$P(x) = x^4 - 4x^3 - 4x^2 + 16x - 8$

$= 0$ is $2 + 2\sqrt{2} + 2\sqrt{3}$.

Reason (R) : $P(x) = (x^4 - 4x^3 + 4x^2) - (8x^2 - 16x + 8)$

$= (x^2 - (2 + 2\sqrt{2})x + 2\sqrt{2})$

$(x^2 - (2 - 2\sqrt{2})x - 2\sqrt{2})$

Sum of absolute value of the roots

$= 2 + 2\sqrt{2} + \sqrt{(2 - 2\sqrt{2})^2 + 4.2\sqrt{2}}$

89. **Assertion (A) :** If x, y, z ∈ R and $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, then the maximum possible value of x – y + z is 4.

Reason (R) : The above equation rearranges as such of three squares equated to zero.

90. **Assertion (A) :** There are only two cubic polynomials $ax^3 + bx^2 + cx + d$, $a \neq 0$ whose coefficients a, b, c, d are a permutation of numbers 0, 1, 2, 3 (without repetition) and having all rational roots.

Reason (R) : The polynomials $x^3 + 3x^2 + 2x$ and $2x^3 + 3x^2 + x$ have three rational roots each.

91. **Assertion (A) :** There is no function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$ whose coefficients are a permutation of 0, 1, 2, 3, 4 (without repetition) and whose all roots are rational.

Reason (R) : f(1) = a + b + c + d + e = 10 cannot be written as a product of 3 integers greater than 1.

92. **Assertion (A) :** If two quadratic function $p(x) = a_1x^2 + b_1x + c_1$ and $q(x) = a_2x^2 + b_2x + c_2$ satisfy p(w) = p(z) and q(w) = q(z) for same w, z ∈ R, $w \neq z$ then $b_1a_1 = b_2a_1$

Reason (R) : $\frac{p(w)-p(z)}{w-z} = a_1(w+z)+b_1 = 0;$

$\frac{q(w)-q(z)}{w-z} = a_2(w+z)+b_2 = 0;$

$w+z = \frac{-b_1}{a_1} = \frac{-b_2}{a_2} \Rightarrow b_1a_1 = b_2a_1.$

93. Assertion (A) : The function $f(x) = x^4 + 4x^2 - x + 6$ cannot be resolved into two quadratic factors with real coefficients.

Reason (R) : In the equation $(x^2+2)^2 + 2 = x$ we observe that a solution to $x^2 + 2 = x$ is a solution of the quadratic equation by substitution of the left hand side into itself.

94. Assertion (A) : There is only one ordered pair (a, b) of real numbers for which $x^2 + ax + b = 0$ has a non real root whose cube is 343.

Reason (R) : $x^3 - 343 = (x-7)(x^2+7x+49)$. Therefore the orderd pair (7, 49) is suitable.

95. Assertion (A) : If P(x) is a quadratic polynomial with real coefficients such that

$P(x+1) . P(x^2-x+1)$ real coefficients such that $P(x+1) . P(x^2-x+1) = P(x^3+1)$ for all real x, then $P(2) = 4$.

Reason (R) : Put $P(x) = ax^2 + bx + c$ into the identity.

$[a(x+1)^2 + b(x+1) + c]\,[a(x^2-x+1)$
$+ b(x^2-x+1) + c] \equiv [a(x^3+1)^2 + b(x^3+1) + c]$

On comparision of coefficients we get

$P(x) = x^2 - \frac{1}{4}x + 1.$

Match the Columns for JEE Advanced

96. Match equations on left with the properties on right.

Column - I	**Column - II**
(A) $a < b < c < d$ and equation is $(x-a)(x-c) + \pi(x-b)(x-d) = 0$	(P) real roots
(B) $a > 0$, $a + b + c < 0$ and equation is $ax^2 + bx + c = 0$	(Q) distinct real roots
(C) b, c, $\in$ I and the equation $x^2 + bx + c = 0$ has rational roots	(R) integral roots
(D) a, b, c, d $\in$ R are in G.P. and equation is $(a^2+b^2+c^2)x^2 + 2(ab+bc+cd)x + b^2+c^2+d^2 = 0$	(S) discriminant ≥ 0

97.

Column-I	**Column-II**
(A) If a, b and c are positive real numbers, then $ax^3 + bx + c = 0$ has	(P) $4a^2 < (2b-c)a$
(B) If $c > 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real root, then (a, b, c $\in$ R)	(Q) $a^2 + a(b+c) > 0$
(C) If the quadratic equation $ax^2 + bx + c = 0$ has real roots and -2 lies between the roots, then (a, b, c $\in$ R)	(R) $a^2 < 2a(b-2c)$
(D) If the quadratic equation $ax^2 + bx + c = 0$ has roots α, β such $\alpha < -2$ and $\beta > 2$ then	(S) $a^2 + a(b+c) < 0$

98. The set of value(s) of $k \in R$ for which

Column - I	Column - II
(A) $kx^2 - (k+1)x + 2k - 1 = 0$ has no real roots	(P) $\{1, -2\}$
(B) $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ for each x	(Q) $(-\infty, -1/7) \cup (1, \infty)$
(C) Sum of the roots of $x^2 + (2 - k - k^2)x - k^2 = 0$ is zero	(R) $\{-4\}$
(D) The roots of $x^2 + (2k-1)x + k^2 + 2 = 0$ are in the ratio 1 : 2	(S) $(2,4)$

99.

Column-I	Column-II
(A) The value of m for which $\dfrac{x^2 - bx}{ax - c} = \dfrac{m-1}{m+1}$ have roots equal in magnitude and opposite in sign is/are	(P) 3
(B) The value of m for which $x^2 - 15 - m(2x - 8) = 0$ has equal roots, is/are	(Q) $\dfrac{a-b}{a+b}$
(C) If α, β are roots of the equation $x^2 - ax + b = 0$, then $(\alpha - a)^{-4} + (\beta - a)^{-4}$ equals	(R) 5
(D) Reciprocal of greatest value of $\dfrac{x+2}{2x^2+3x+6}$ is	(S) $\dfrac{a^4 - 4a^2b + 2b^2}{b^4}$

100. Let $a, b, c \in R$ be such $b^2 - ac \geq 0$. the equation $ax^2 + 2bx + c = 0$ has

Column - I	Column - II
(A) two positive roots	(P) a, –c are of the same sign
(B) two negative roots	(Q) a, –b, c are of the same sign
(C) one positive and one negative roots	(R) a, b, c are of the same sign
(D) equal roots	(S) a, b, c are in G. P.

Review Exercises for JEE Advanced

1. If the equation $ax^2 + 2bx + c = 0$ has real roots, a, b and c being real numbers and if m and n are real numbers such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

2. If the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + c^4 + d^4 = 0$ are real then prove that they must be equal.

3. Prove that if $x^2 + px - q$ and $x^2 - px + q$ both factorise into linear factors with integral coefficients, then the positive integers p and q are respectively the hypotenuse and area of a right triangle with sides of integer length.

4. If $2(a + b + c) = \alpha^2 + \beta^2 + \gamma^2$, and the roots of $x^2 + \alpha x - a = 0$ are β, γ and the roots of $x^2 + \beta x - b = 0$ are γ, α, show that the equation whose roots are α, β is $x^2 + \gamma x - c = 0$.

5. If α, β are the roots of equation $x^2 - p(x+1) - c = 0$, show that $(\alpha + 1)(\beta + 1) = 1 - c$. Hence prove that $\dfrac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \dfrac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c} = 1$

6. If $\beta + \cos^2\alpha, \beta + \sin^2\alpha$ are the roots of $x^2 + 2bx + c = 0$ and $\gamma + \cos^4\alpha, \gamma + \sin^4\alpha$ are the roots of $x^2 + 2Bx + C = 0$ then prove that $b^2 - B^2 = c - C$.

7. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ac)x + c^2 - ab = 0$ are equal if either $b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

8. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the other root of the first then prove that $(q - b)^2 = bq(p - a)^2$.

9. If the equation $x^2 - 2px + q = 0$ has two equal roots, then prove that the equation $(1 + y)x^2 - 2(p + y)x + (q + y) = 0$ will have roots real and distinct only when y is negative and p is not unity.

10. If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, then show that its other root is a root of $x^2 + (2a - m)x + a^2 - am + n = 0$.

11. Find all real values of parameter p for which the least value of the quadratic expression $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \le x \le 2$ is equal to 3.

12. If $\dfrac{x^2 + ax + 3}{x^2 + x + a}$ takes all real values for possible real values of x then prove that $4a^3 + 39 < 0$.

13. Find the range of values of a for which all the roots of the equation $(a - 1)(1 + x + x^2)^2 = (a + 1)(1 + x^2 + x^4)$ are imaginary.

14. If the quadratic function, $y = (\cot\alpha)x^2 + 2(\sqrt{\sin\alpha})x + \frac{1}{2}\tan\alpha$, $\alpha \in (0, 2\pi) - \{\pi/2, \pi\}$, can take negative values for all $x \in R$, then find the interval in which α lies.

15. Find the value of a for which the range of the function $y = \dfrac{x - 1}{a - x^2 + 1}$ does not contain any value belonging to the interval $\left[-1, -\frac{1}{3}\right]$.

16. Find the values of a for which the inequality is satisfied $25^x + (a + 2)5^x - (a + 3) < 0$ for atleast one x.

17. Find the integral values of 'a' for which the equation $x^4 - (a^2 - 5a + 6)x^2 - (a^2 - 3a + 2) = 0$ has real roots only.

18. Find the condition on a, b, c such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

19. For what values of $a \in R$ does the equation $ax^2 + x + a - 1 = 0$ possess two distinct real roots x_1 and x_2 satisfying the inequality $\left|\dfrac{1}{x_1} - \dfrac{1}{x_2}\right| > 1$?

20. Find all the values of the parameter a for which the inequality $4^x - a \cdot 2^x - a + 3 \le 0$ is satisfied by atleast one real x.

21. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and α^4 and β^4 are the roots of the equation $\ell x^2 + mx + n = 0$, then prove that the roots of the equation $a^2\ell x^2 - 4ac\ell x + 2c^2\ell + a^2m = 0$ are always real and opposite in sign. (α, β are real and different)

22. If the roots of the equation $\dfrac{1}{x + p} + \dfrac{1}{x + q} = \dfrac{1}{r}$ are equal in magnitude but opposite in sign, show that $p + 1 = 2r$ and that the product of the roots is equal to $-\dfrac{p^2 + q^2}{2}$.

23. Find all values of a for which the inequation $4^{x^2} + 2(2a + 1)2^{x^2} + 4a^2 - 3 > 0$ is satisfied for any x.

24. For what values of a, the equation $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ has $(a \in R)$
(a) both roots smaller than 2
(b) both roots greater than 2
(c) both roots lying in the interval (2, 3)
(d) exactly one root lying in the interval (2, 3)
(e) one root smaller than 1, and other root greater than 1
(f) atleast one root lying in the interval (2, 3)
(g) one root less than 2 and other greater than 3.

25. For what values of $a \in R$ does the equation $x^2 + 1 = x/a$ possess two distinct real roots x_1 and x_2 satisfying the inequality $|x_1^2 - x_2^2| > 1/a$?

26. For what values of the parameter a is the inequality

$[a^3 + (1 - \sqrt{2})a^2 - (3 + \sqrt{2})a + 3\sqrt{2}]x^2$

$+ 2(a^2 - 2)x + a > -\sqrt{2}$ satisfied for any $x > 0$?

27. For what values of the parameter p the equation $x^4 + 2px^3 + x^2 + 2px + 1 = 0$ has atleast two distinct negative roots.

28. If α, β, γ are the roots of $x^3 + qx + r = 0$, prove that the equation whose roots are

$\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

is $r^2(z + 1)^3 + q^3(z + 1) + q^3 = 0$.

29. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with this cubic, prove that either $a = b = 0$ or $a = b = 3$, or a, b are the roots of $z^2 + z + 2 = 0$.

30. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of equation $x^n + nax - b = 0$, show that

$(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) .... (\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$

31. Find the integral part of the greatest root of equation $x^3 - 10x^2 - 11x - 100 = 0$

32. Two roots of the equation

$x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$

are of the form $\alpha + i\beta$, $\beta + i\alpha$. Find all the roots of the equation.

33. If α, β, γ are the roots of the cubic $x^3 - px^2 + qx - r = 0$ find the cubic equations whose roots are

(i) $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$

(ii) $(\beta + \gamma - \alpha), (\gamma + \alpha - \beta), (\alpha + \beta - \gamma)$

Also find the value of $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$

34. If the equation $x^4 - 4x^3 + ax^3 + bx + 1 = 0$ has four positive roots, then find a and b.

35. If the equations $x^2 - ax + b = 0$ & $x^3 - px^2 + qx = 0$, where $b \neq 0, q \neq 0$ have common roots & the second equation has two equal roots, then prove that $2(q + b) = ap$.

Target Exercises for JEE Advanced

1. Let $p(x) = ax^2 + bx + c$ be such that p(x) takes real values for real values of x and non-real values for non-real values of x. Prove that $a = 0$.

2. For what values of the quantity h is the polynomial

$x^4 - 2^{tanh} . x^2 + (\cos h + \cos 2h)x + 2^{tanh-2}$

the square of the quadratic trinomial with respect to x ?

3. Find the values of a for which the equation

$(x^2 + x + 2)^2 - (a - 3)(x^2 + x + 2)(x^2 + x + 1) + (a - 4)(x^2 + x + 1)^2 = 0$ has atleast one real root.

4. Find all values of α for which the solutions of the system of inequalities

$x^2 + 6x + 7 + \alpha \leq 0$, $x^2 + 4x + 7 \leq 4\alpha$ form an interval of length unity on the number axis.

5. For real values of x, if the expression $\frac{(ax - b)(dx - c)}{(bx - a)(cx - d)}$ assumes all real values then prove that $(a^2 - b^2)$ and $(c^2 - d^2)$ must have the same sign.

6. Find a for which the range of the function

$y = \frac{x - 1}{1 - x^2 - a}$ does not contain any value from the interval $[-1, 1]$

7. For what values of 'k', the inequality $\frac{x^2 + k^2}{k(x + 6)} \geq 1$ satisfied for all $x \in (-1, 1)$.

8. Let A, B, C be three angles such that $A = \pi/4$ and tan B. tan C = p find all possible values of p such that A, B, C are the angles of a triangle.

9. Find all values of α for which the system of inequalities

$x^2 + 4x + 3 \le \alpha$, $x^2 - 2x \le 3 - 6a$ has a unique solution.

10. Find all real values of a for which the equation $\sqrt{(x-a)}\,[x^2 + (1 + 2a^2)x + 2a^2] = 0$ has only two distinct roots.

11. Find the value of a for which the equation $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$
(a) has no solution
(b) has one solution
(c) has two solutions
(d) has three solutions

12. Find all values of k for which all solutions of the inequation $kx^2 - 2(k^2 - 3)x - 12 \ge 0$ are the solutions of the inequation $x^2 - 49 \ge 0$

13. Find all the values of a for which from the inequality $x^2 - a(1 + a^2)x + a^4 < 0$ follows the inequality $x^2 + 4x + 3 > 0$.

14. Find all the values of a for which from the inequality $0 \le x \le 1$ follows the inequality $(a^2 + a - 2)x^2 - (a + 5)x - 2 \le 0$

15. Find the values of a for which the range of function $y = \dfrac{x-1}{a+x^2}$ contain the interval [0, 1].

16. Find the least possible value of the expression $\dfrac{2x^4 + 4x^3 + 9x^2 - 4x + 2}{(x^2+1)^2}$ for real values of x.

17. Find the set of values of k for which any solution of the inequality $\dfrac{\log_2(x^2 - 5x + 6)}{\log_2(2x)} < 1$ is also a solution of the inequality $x^2 - 7kx + 2k - 6 \le 0$.

18. If $f(x) = x^3 + bx^2 + cx + d$ and f(0), f(–1) are odd integers, prove that f(x) = 0 cannot have all integral roots.

19. If the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive roots, then find the minimum value of pr.

20. If equation $ax^3 + 2bx^2 + 3cx + 4d + 0$ and $ax^2 + bx + c = 0$ have a non-zero common root, then prove that $(c^2 - 2bd)(b^2 - 2ac) \ge 0$.

21. Prove that the roots of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot all be real if $2a^2 < 5b$.

22. If $(n-1)x^2 - 2(a_n - a_1)x + a_1^2 + 2a_3^2 + \ldots + 2a_{n-1}^2 + a_n^2 = 2(a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n)$. show that $a_1, a_2, \ldots, a_n$ are in A.P. with common difference x.

23. If a, b, c be the three possible values of $n \in N$ for which the polynomial $1 + x^2 + x^4 \ldots + x^{2n-2}$ is divisible by $1 + x + x^2 + \ldots + x^{n-1}$, then show that the roots of the quadratic equation $ax^2 + bx + c = 0$ are not rational.

24. If the equations $ax^3 + bx + c = 0$ and , $a'x^3 + b'x + c' = 0$ have a common root then prove that
$$a(ca'-c'a)^3 + b(ab'-a'b)^2(ca'-c'a) + c(ab'-a'b)^3 = 0.$$

25. If $(x^2 - 7x + 12).f(x) = (x^2 + 7x + 12).g(x)$ then prove that $h(x) = f(x)\,.\,g(x) + x^4 - 25x^2 + 144$ has four real roots. Also find them.

26. Show that $ax^4 + bx^3 + \left(a + \dfrac{b^2}{4a}\right)x^2 + dx + e$ will have the same sign as a if $4ae > d^2$.

27. Show that $\dfrac{x^3 - 3x + 1}{x(x-1)}$ is unaltered by substituting either $\dfrac{1}{1-x}$ or $1 - \dfrac{1}{x}$ for x. Hence or otherwise find all the roots of $\dfrac{x^3 - 3x + 1}{x(x-1)} = \dfrac{c^3 - 3c^2 + 1}{c(1-c)}$.

28. Prove that the necessary and sufficient conditions that the roots of $\lambda(ax^2 + bx + c) + \mu(a'x^2 + b'x + c') = 0$ may be real all real vaues of λ and μ are that $b^2 - 4ac > 0$ and $(bc' - b'c)(ab' - a'b) - (ca' - c'a)^2 > 0$.

29. Find all values of k for each of which there is atleast one common solution of the inequalities $x^2 + 4kx + 3k^2 > 1 + 2k$, $x^2 + 2kx \le 3k^2 - 8k + 4$.

30. Find all real values of the quantity h for which the equation $x^4 + (h - 1)x^2 + x^2 + (h - 1)x + 1 = 0$ possesses not less than two distinct negative roots.

31. For what values of a does the equation $\log_3(9^x + 9a^3) = x$ possess two solutions?

32. Find the real values of the parameter a for which every solution of the inequality $\log_{1/2} x^2 \ge \log_{1/2}(x + 2)$ is a solution of the inequality $49x^2 - 4a^4 \le 0$.

33. For what values of 'p' does the equation $p\,2^x + 2^{-x} = 5$ possess a unique solution.

34. For what real values of a is the sum of roots of the equation $\left(\frac{1}{x} + \frac{1}{a} - \frac{1}{a^2} = \frac{1}{-a^2 + a + x}\right)$ smaller than $a^3/10$?

35. Prove that for every a > 0 the inequality $\sqrt{a + \sqrt{a + \sqrt{a + ... + \sqrt{a}}}} < \frac{1 + \sqrt{4a + 1}}{2}$ (the left side contains an infinite number of radicals) holds true.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. If : $2 + i\sqrt{3}$ a root of the equation $x^2 + px + q = 0$ where p and q are real, then (p, q) = (............). **[IIT - 1982]**

2. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \ne b)$ have a common root, then the numerical value of a + b is............. **[IIT - 1986]**

B. True / False :

3. If a < b < c < d, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct. **[IIT - 1984]**

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \ne 0$, then $P(x)\,Q(x) = 0$ has at least two real roots. **[IIT - 1985]**

C. Multiple Choice Question with ONE correct answer :

5. If l, m, n are real, $l \ne m$, then the roots by the equation: $(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$ are **[IIT - 1979]**

(A) Real and equal (B) Complex
(C) Real and unequal (D) None of these

6. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always. **[IIT - 1979]**

(A) non negative (B) zero
(C) non positive (D) None of these

7. Let a > 0, b > 0 and c > 0. Then the roots of the equation $ax^2 + bx + c = 0$ **[IIT - 1979]**

(A) are real and negative
(B) have negative real parts
(C) are imaginary
(D) none of these

8. Both the roots of the equation $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always **[IIT - 1980]**

(A) positive (B) real
(C) negative (D) None of these

9. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then **[IIT - 1980]**

(A) $a^2 + c^2 = -ab$ (B) $a^2 - c^2 = -ab$
(C) $a^2 - c^2 = ab$ (D) None of these

10. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is **[IIT - 1982]**

(A) 4 (B) 1 (C) 3 (D) 2

11. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is **[IIT - 1982]**

(A) $-4 < x \le 0$ (B) $0 < x < 1$
(C) $-100 < x < 100$ (D) $-\infty < x < \infty$

12. If $a^2 + b^2 + c^2 = 1$, then ab + bc + ca lies in the interval **[IIT - 1984]**

(A) [1/2, 2] (B) [–1, 2]
(C) [–1/2, 1] (D) [–1, 1/2]

13. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always **[IIT - 1989]**

(A) two real roots
(B) two positive roots
(C) two negative roots
(D) one positive and one negative roots

14. Let a, b, c be real numbers $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies **[IIT - 1989]**

(A) $\gamma = \frac{\alpha + \beta}{2}$ (B) $\gamma = \alpha + \frac{\beta}{2}$
(C) $\gamma = \alpha$ (D) $\alpha < \gamma < \beta$.

15. Let α, β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are **[IIT - 1992]**
(A) a, c (B) b, c (C) a,b (D) a + c, b + c

16. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real less than 3; then **[IIT - 1999]**
(A) $a < 2$ (B) $0 \leq a \leq 3$
(C) $3 < a \leq 4$ (D) $a > 4$

17. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then **[IIT - 2000]**
(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta < |\alpha|$
(C) $\alpha < \beta < 0$ (D) $\alpha < 0 < |\alpha| < \beta$

18. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has **[IIT - 2000]**
(A) both roots in (a, b)
(B) both roots in $(-\infty, a)$
(C) both roots in $(a, +\infty)$
(D) one root in $(-\infty, a)$ and the other in $(b, +\infty)$

19. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the root is square of the other, then p is equal to **[IIT - 2000]**

(A) 1/3 (B) 1 (C) 3 (D) 2/3

20. For all 'x', $x^2 + 2ax + 10 - 3a > 0$, then the interval in which 'a' lies is **[IIT - 2004]**
(A) $a < -5$ (B) $-5 < a < 2$
(C) $a > 5$ (D) $2 < a < 5$

21. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is **[IIT - 2004]**
(A) $p^3 - q(3p - 1) + q^2 = 0$
(B) $p^3 - q(3p + 1) + q^2 = 0$
(C) $p^3 + q(3p - 1) + q^2 = 0$
(D) $p^3 + q(3p + 1) + q^2 = 0$

22. If p, q, r are +ve and are in A.P., the roots of quadratic equation $px^2 + qx + r = 0$ are all real for **[IIT - 1994]**

(A) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (B) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

(C) all p and r (D) no p and r

23. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is **[IIT - 1994]**
(A) 15 (B) 9 (C) 7 (D) 8

24. Let a, b, c be the sides of a triangle where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ real, then **[IIT - 2006]**

(A) $\lambda < \frac{4}{3}$ (B) $\lambda < \frac{5}{3}$

(C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

25. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}$; 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is **[IIT - 2007]**

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

26. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ as its roots is **[IIT - 2010]**

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

27. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT - 2011]**

(A) 1 (B) 2

(C) 3 (D) 4

28. A Value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0$$

have one root in common is **[IIT - 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$

(C) $i\sqrt{5}$ (D) $-\sqrt{2}$

29. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2 + \left(\sqrt{1+a}-1\right) + \left(\sqrt[6]{1+a}-1\right) = 0$

where $a > -1$.

Then $\lim_{a \to 0+} \alpha(a)$ and $\lim_{a \to 0+} \beta(a)$ are **[IIT - 2012]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and 1

(C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

D. Multiple Choice Question with ONE or MORE THAN ONE correct answer :

30. For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided **[IIT - 1984]**

(A) $a > b > c$ (B) $a < b < c$

(C) $a > c > b$ (D) $a < c < b$

31. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$ then a, b, c, d **[IIT - 1987]**

(A) are in A.P. (B) are in G.P.

(C) are in H.P. (D) satisfy ab = cd

E Subjective Problems:

32. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s.

Deduce the condition that the equations have a common root. **[IIT - 1979]**

33. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n-th power of the other, then show that $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$ **[IIT - 1983]**

34. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$. **[IIT - 1995]**

35. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. **[IIT - 1996]**

36. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$. **[IIT - 1997]**

37. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$ $(A \neq 0)$ for some constant δ, then prove that

$$\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$$ **[IIT - 2000]**

38. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. **[IIT - 2001]**

39. If $x^2 + (a - b)x + (1 - a - b) = 0$ where $a, b \in R$ then find the values of a for which equation has unequal real roots for all values of b. **[IIT - 2003]**

40. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d then the value of $a + b + c + d$, when $a \neq b \neq c \neq d$, is. **[IIT - 2006]**

F. Assertion A and Reason R :

41. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$. **[IIT - 2008]**
Assertion (A): $(p^2 - q)(b^2 - ac) \geq 0$
Reasons (R): $b \neq pa$ or $c \neq qa$

G Integer answer type

42. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have value at least 4, is. **[IIT - 2009]**

Answers

Practice Problems

1. m, $\frac{1-m^2}{2m}$ **2.** 1, $\frac{a-b}{b-c}$
3. –1, 4 **4.** –1/3, 1/2
5. –9, 2

Practice Problems

1. $a = -2, c = -4$ **2.** $x^2 + (31/36)x - (449/216) = 0$
3. $4x^2 - 5x - 6$ **4.** (a) ϕ (b) $-\frac{8}{9}$ (c) $\frac{-3}{11}$
5. (a) 15 (b) – 22 (c) 127
7. $a_1 = -3/2, a_2 = 6$ **8.** $a_1 = 3/2, a_2 = 3$
10. –3, –4 **12.** $b = -13$
14. $a^2l^2x^2 - ablmx + (b^2 - 2ac)\ln + (m^2 - 2ln)ac = 0$

Practice Problems

1. 1 **2.** $p(x) = x^2 + c$
3. $a = 3, b = -3, c = 1.$ **4.** $a = 5, b = -3, c = 1.$
5. $x \in R$
6. $x = ab + bc + ca$; If $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$ the given equation becomes an identity & is true for all $x \in R$
7. $8x^2 - 10x - 61 = 0$
9. (i) $x^2 - 6x + 11 = 0$ (ii) $3x^2 - 2x + 1 = 0$
10. (i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$, (ii) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

Practice Problems

1. – 43, – 29/14, 14
2. $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$
3. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$
4. (i) $\frac{p(p^2 - 4q)(p^2 - q)}{q}$ (ii) $\frac{p^4 - 4p^2q + 2q^2}{q^4}$
5. $\frac{b^3 - 3abc}{a^3c^3}$ **7.** – 160
8. $x^2 - 3x + 2 = 0$ **9.** $x^2 + 31x + 112 = 0$

Practice Problems

1. $a = 4$ **2.** 1, – 1/3
4. (a) $a \geq -81/4$, (b) $a \leq 2$, (c) all real a
9. real and distinct.

Practice Problems

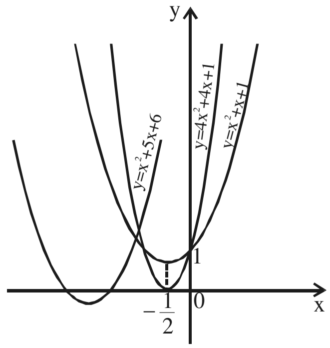
1. $k^2 + k, k \in W$ **5.** $\{-2, 0\}$
6. –4, –3

Practice Problems

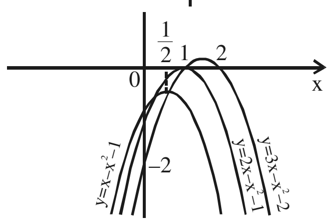
7. $n\pi + (-1)^n \frac{\pi}{6}, n \in I$ **8.** $\lambda = 3/2, \mu = -6$

Practice Problems

1.



2.



3. {–4, –3, 3, 4}

4. (a) $(-\infty, 1.5)$, (b) $(-2, \infty)$

5. (a) $(-1.5, \infty)$, (b) $(-\infty, 2)$

6. $[5, \infty)$

7. 4

8. (i) $a < 0, b < 0, c > 0, D > 0$,

(ii) $a > 0, b > 0, c > 0, D = 0$,

(iii) $a < 0, b > 0, \quad c < 0, \; D < 0$.

9. Negative

12. positive

Practice Problems

1. (i) $[-1, 4]$, (ii) $\{-1/2\}$

2. (A) All reals,

(B) $x > 3$ or $x < 1$

(C) $-1-\sqrt{13} < x < -1+\sqrt{13}$,

(D) $-1 \le x \le 8/3$

3. (i) $[-4, -1) \cup (3, \infty)$, (ii) {4}, (iii) (6, 8]

4. $\left(-\infty, -\dfrac{a-\sqrt{a^2-4a}}{2}\right) \cup \left(\dfrac{\sqrt{a^2-4a}-a}{2}, \infty\right)$, $a \in (-\infty, 0) \cup (4, \infty)$; R , $a \in (0, 4)$; $R - \left\{-\dfrac{a}{2}\right\}$, $a \in \{0, 4\}$

5. 12, 13

6. $k \le -7/9$, $k \ge 1$

7. [0, 1/2)

8. $k = 3$

9. For all $a \in (5/3, +\infty)$.

10. $(-\infty, 6)$

11. $(-6, 3)$

12. 3

13. $(-\infty, 0) \cup \left(\dfrac{9}{2}, \infty\right)$

15. (a) Not for any a, (b) for any a, (c) $a \ne 0$, (d) for any a, (e) $a = 0$, (f) $a \ne 0$

Practice Problems

1. (i) $-\dfrac{3}{2}, \infty$, (ii) $-1, \infty$ (iii) $-\dfrac{3}{2}, 23$

2. (a) $y_{min} = y(1) = 7$, $y_{max} = y(2) = 15$
(b) $y_{max} = y(2) = -14$, $y_{min} = y(3) = -29$
(c) $y_{max} = y(-1) = 8$, $y_{min} = y(1) = 4$
(d) $y_{min} = y(0) = -1$, $y_{max} = y(3) = 8$.

3. $\dfrac{25}{8}$.

4. $-3/8$, $33/8$

5. –2

6. $y \in \left(-\infty, -\dfrac{7}{8}\right]$

7. –7, 1

Practice Problems

1. $y \in (-\infty, -2] \cup (1, \infty)$

9. $-12 < a < 2$

10. $m < \dfrac{11}{4}$

Practice Problems

1.(i) $(3x+4y-1)(x-y+3)$ (ii) $(3x-4y+2)(x+2y-3)$

2.(i) $\lambda = 1$, $(2x + y - 2)(x - y + 1)$; $\lambda = -7/8$, $(x - 7y + 4)(x - y - 4)/8$
(ii) $\lambda = -1$, $(y - 1)(y - 2)$; $\lambda = -2$, $-(x + y)(x - y)$;

$\lambda = -\frac{10}{9}$, – (x + 3y + 4) (x – 3y – 4)/9.

3. – 5/2 **4.** 0, – 17/36

Practice Problems

1. $\left[3, \frac{15}{4}\right]$ **2.** (0, 4)

3. $\frac{a-b}{a+b}$ **4.** 2 < m < 8/3

5. (i) (a) (0, 3), (b) {0, 3},
(c) $(-\infty, 0) \cup (3, \infty)$, (d) $(-\infty, -1) \cup [3, \infty)$, (e) ϕ

(ii) $\left(-1, -\frac{1}{8}\right)$ (iii) $\left\{-\frac{1}{3}, 0, 3\right\}$ (iv) $\left\{-\frac{1}{3}\right\}$

(v) $(-\infty, -1) \cup \left[-1, -\frac{1}{2}\right] \cup [3, \infty)$

(vi) $\left(-1, -\frac{1}{8}\right)$

(vii) {0} (viii) $\left(-1, -\frac{1}{3}\right)$

(ix) $\left(-\frac{1}{3}, -\frac{1}{8}\right)$

6. ϕ **7.** ϕ

8. a < –2 **9.** $\left(0, \frac{1}{8}\right)$

10. $2\sqrt{2} \le a < \frac{11}{3}$

11. (i) $[4, \infty)$ (ii) (–1, 0) (iii) (–1, 0]
(iv) $[3, \infty)$ (v) $(-\infty, -1) \cup [1, \infty)$ (vi) ϕ (vii) ϕ

13. $\left[\frac{1}{2}, \infty\right)$

Practice Problems

2. $(1, \infty)$ **3.** $\left(\frac{5}{2}, \infty\right)$

4. a < 3

6. (a) (0, 3), (b) ϕ, (c) $(-\infty, -1)$, (d) $(1, \infty)$

7. $[1, \infty)$

Practice Problems

5. p = –2, q = –4 **6.** $7x^2 - 11x - 6$

Practice Problems

1. $-\sqrt{5} - 3, 2, 3$ **2.** $x^4 - 16x^2 + 8x - 1 = 0$

3. 1 ± 2i and 2 ± i **4.** $2 \pm \sqrt{3}, -3 \pm \sqrt{2}$

5. $x^4 - 16x^2 + 4 = 0$

8. $(x^2+x+1)(x^2-x+1)(x^2+\sqrt{3}x+1)$
$(x^2-\sqrt{3}x+1)$

9. –2, 1/2, 3 **10.** 2

Practice Problems

1. $2 \pm \sqrt{2}$ **2.** $-2, -\frac{1}{2}, \frac{3 \pm \sqrt{5}}{2}$

3. $\frac{-11 \pm \sqrt{97}}{6}$ **4.** 4, 2

5. $-\frac{1}{12}, \frac{1}{2}$ **6.** $\frac{1 \pm \sqrt{5}}{2}$

Practice Problems

1. –1, 2, –3/2 **2.** –1, 1, 3, 5

3. 1, 3, 5, 7 **4.** $-\frac{3}{2}$, –2, 4

5. $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$ **6.** 4/3, 3/2, $1 \pm \sqrt{2}$

8. 6, 2, 2/3

Practice Problems

1. $y^3 - 8y^2 + 197 - 15 = 0$

2. 4, 2, 4/3 **3.** $x^3 - 2x^2 + 5x - 11 = 0$

4. 6

Practice Problems

1. (i) $\frac{1}{a^2}(b^2 - 2ac)$ (ii) $\frac{-c}{a}$ (iii) $\frac{3ad - bc}{a^2}$

2. (i) $3p^3 - 16pq + 64r$.
(ii) $(q^3 - 4pqr + 8r^2)/r^3$.
(iii) $(q^3 - p^3r)/r^4$.

3. $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}$

4. $\frac{3}{2}, -\frac{3}{4}\left(1 \pm \sqrt{5}\right)$

5. (a) $\frac{q^2 - 2pr}{r^2}$ (b) $\frac{p^2 - 2q}{r^2}$

6. (a) $-6q$ (b) q/r

Objective Exercise

1. A	**2.** A	**3.** D	**4.** A	**5.** A
6. D	**7.** C	**8.** C	**9.** D	**10.** A
11. D	**12.** C	**13.** D	**14.** C	**15.** C
16. A	**17.** B	**18.** B	**19.** C	**20.** B
21. B	**22.** D	**23.** B	**24.** B	**25.** C
26. B	**27.** A	**28.** B	**29.** A	**30.** A
31. B	**32.** B	**33.** B	**34.** D	**35.** B
36. B	**37.** C	**38.** B	**39.** A	**40.** B
41. C	**42.** C	**43.** D	**44.** A	**45.** A
46. A	**47.** A	**48.** C	**49.** B	**50.** C
51. A	**52.** ABC		**53.** CD	**54.** BC
55. AC	**56.** ABCD		**57.** CD	**58.** BCD
59. AB	**60.** ABC	**61.** ABC	**62.** ABC	**63.** BC
64. ABC		**65.** AC	**66.** BD	**67.** BD
68. BD	**69.** BC	**70.** ABC	**71.** B	**72.** C
73. B	**74.** B	**75.** A	**76.** C	**77.** C
78. B	**79.** C	**80.** C	**81.** D	**82.** B
83. C	**84.** B	**85.** C	**86.** C	**87.** C
88. A	**89.** B	**90.** B	**91.** A	**92.** A
93. D	**94.** A	**95.** C		

96. (A) – PQS, (B) – PQS, (C) – PRS, (D) – PS
97. (A) – Q, (B) – Q, (C) – P, (D) – PRS
98. (A) – QRS, (B) – S, (C) – P, (D) – R
99. (A) – Q, (B) – PR, (C) – S, (D) – P
100. (A) – Q, (B) – R, (C) – P, (D) – S

Review Exercises for JEE Advanced

1. 0 **5.** 1

11. $1 - \sqrt{2}, 5 + \sqrt{10}$ **13.** $(-2, 2)$

14. $\left(\frac{5\pi}{6}, \pi\right)$ **15.** $a < -\frac{1}{4}$

16. $R - \{-4\}$ **17.** $a = 1, 2$

18. $(4bc + ad)^2 = \frac{9}{2}(bd + 4c^2)(b^2 - ac)$

19. $a \in (0, 1) \cup (1, 6/5)$ **20.** $a \geq 2$

23. $a \in (-\infty, -1) \cup [\sqrt{3}/2, \infty)$

24. (a) $a \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right)$

(b) $a \in \left(\frac{7+\sqrt{33}}{2}, \infty\right)$

(c) $a \in \left(-\frac{7}{2}, -\frac{5}{2}\right)$

(d) $a \in \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$

(e) $a \in (0, 3)$

(f) $a \in \left(-\frac{7}{2}, -\frac{5}{2}\right) \cup \left(\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2}\right)$

(g) $a \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$

25. $a \in (-1/2, 0) \cup (0, \sqrt{5}/5)$.

26. $(-\sqrt{2}, 1) \cup [\sqrt{2}, +\infty)$

27. $p \in \left(\frac{3}{4}, \infty\right)$ **31.** 11

32. $2 \pm i, 1 \pm 2i$

33. (i) $ry^3 - q(r + 1)y^2 + p(r + 1)^2 y - (r + 1)^3 = 0$;
(ii) $y^3 - py^2 + (4q - p^2)y + (8r - 4pq + p^3) = 0$;
and $4qp - p^3 - 8r$

34. $a = 6, b = -4$

Target Exercises for JEE Advanced

2. $h = (2k+1)\pi, k \in I$ **3.** $5 < a \le 19/3$

4. $\alpha = 1,\ \alpha = \frac{7}{4}$ **6.** no value of a

7. $k \in \left[\frac{7+3\sqrt{5}}{2}, \infty\right)$

8. $p \in (-\infty, 3-2\sqrt{2}] \cup [3+2\sqrt{2}, \infty)$

9. $\alpha = 1, \alpha = 0$

10. $x_1 = a, x_2 = -1$ for $a \in (-\infty, -1)$; $x_1 = a, x_3 = -2a^2$ for $a \in (-1/2, 0)$

11. (a) $a \in (-\infty, -1) \cup (5/4, \infty)$,
(b) $a = -1$,
(c) $a \in (-1, 1) \cup \{5/4\}$,
(d) $a = 1$

12. $k \in (0, 7/6]$ **13.** $a \le -3, a \ge -1.$

14. $-3 \le a \le 3$ **15.** $-\infty < a < -1$ or $-1 < a < \frac{5}{4}$

16. $\frac{6}{5}$ **17.** $[3/4, 3]$

19. 80 **25.** $x = \pm 3, \pm 4$

27. $\frac{1}{c}, \frac{c}{c-1}, 1-c$ **29.** $k < \frac{1}{2}, k > \frac{3}{2}$

30. $h > \frac{5}{2}$ **31.** $0 < a < \frac{1}{\sqrt[3]{36}}$

32. $a \le -\sqrt{7}, a \ge \sqrt{7}$ **33.** $p \in (-\infty, 0] \cup \left\{\frac{25}{4}\right\}$

34. $a \in (0, 1) \cup (1, 5-\sqrt{15}) \cup (5+\sqrt{15}, \infty)$

Previous Year's Questions (JEE Advanced)

1. –4, 7	**2.** 1	**3.** T	**4.** T	**5.** C
6. A	**7.** B	**8.** B	**9.** D	**10.** A
11. D	**12.** C	**13.** A	**14.** D	**15.** C
16. A	**17.**B	**18.** D	**19.** C	**20.** B
21. A	**22.** AB	**23.** C	**24.** A	**25.** D
26. B	**27.** C	**28.** B	**29.** B	
30. CD	**31.** B			

32. $q(r-p)^2 - p(r-p)(s-q) + (s-q)^2$; $(q-s)^2 = (r-p)(ps-qr)$

33. $\beta \in \left(-\infty, \frac{1}{3}\right], \in \left[-\frac{1}{27}, \infty\right)$

34. $\alpha^2\beta, \alpha\beta^2$ **35.** $a > 1$ **36.** 1210

37. B **38.** 2